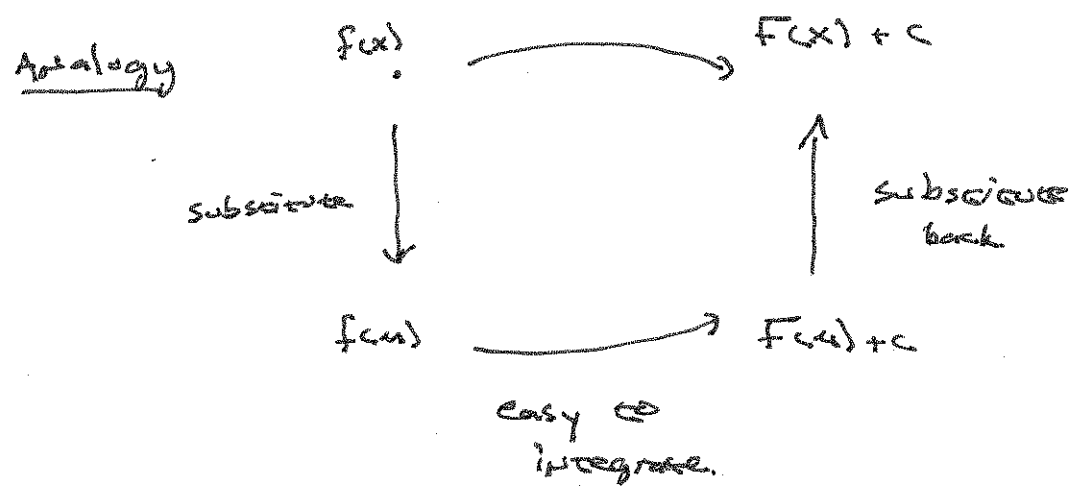


7.1: Defn. of the Laplace Transform

Technique when the D.E. has a driving force up to and including piecewise or sawtooth fcts. It is especially nice for IVP



Named for Pierre-Simon Laplace (1749-1827)

French mathematician
 came from a farming fam. No educated family
 Laplace turned rich teens to pay for school.
 Two big fields included mathematical astronomy
 and probability as well as DE.
 Elected to the French academy @ 23.
 survived the French rev.
 arrogant ... politically "flexible."

Def. The Laplace Transform

Let f be a fct. defined for $t \geq 0$.

Then the integral

$$\mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt$$

is the Laplace Transform of f , provided the integral converges.

Notice $\mathcal{L}\{f(t)\}$ is a fct of s .

Notation: $\mathcal{L}\{f(t)\} = F(s)$

$$\mathcal{L}\{g(t)\} = G(s)$$

$$\mathcal{L}\{y(t)\} = Y(s)$$

ex1: $\mathcal{L}\{1\} = \frac{1}{s}$

ex2: $\mathcal{L}\{t^2\} = \frac{2}{s^3}$

ex3: $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

ex4: $\mathcal{L}\{\cos kt\} = \frac{s}{s^2+k^2}$

Important: \mathcal{L} is linear.

$$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha \mathcal{L}\{f(t)\} + \beta \mathcal{L}\{g(t)\} = \alpha F(s) + \beta G(s)$$

ex 5: $\mathcal{L}\{f(t)\}$ where $f(t) = \begin{cases} 0 & 0 \leq t < \pi/2 \\ \cos t & t \geq \pi/2 \end{cases}$

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \int_0^{\infty} e^{-st} f(t) dt \\ &= \int_0^{\pi/2} e^{-st} \cdot 0 dt + \int_{\pi/2}^{\infty} e^{-st} \cos t dt \\ &= 0 + \frac{e^{-st}}{s^2 + 1} (-s \cos t + \sin t) \Big|_{\pi/2}^{\infty} \\ &= \frac{e^{-\frac{\pi}{2}s}}{s^2 + 1} \left(\underbrace{-s \cos \frac{\pi}{2}}_1 + \underbrace{s \cos \frac{\pi}{2}}_0 \right) \\ &= \frac{e^{-\frac{\pi}{2}s}}{s^2 + 1} \end{aligned}$$

so when does the Laplace transform exist?

Thm: Sufficient conditions for the existence of \mathcal{L} .

If f is piecewise cont. on $[0, \infty)$ and of exponential order, then $\mathcal{L}\{f(t)\}$ exists for $s > c$.

A fct. f is piecewise continuous on $[0, \infty)$ if in any interval $a \leq t \leq b$ there are at most a finite number of points $t_k, k=1, 2, \dots, n$ w/ $(t_{k-1} < t_k)$ at which f has finite discontinuities and is cont on each open interval (t_{k-1}, t_k) .

A fct. f is said to be of exp. order if $\exists c, M > 0$, and $T > 0$ s.c. $|f(t)| \leq M e^{ct}$ for all $t > T$.