7.1: Dfn. of the Laplace Transform

Technique when the D.E. has a driving fct up to and including piecewise or sawtooth fcts. It is especially nice for IVP

Analogy

$$\begin{tikzcd}
f(x) \arrow[r] \arrow[d, "\text{substitute}"'] & F(x)+C \\
f(u) \arrow[r, "\text{easy to integrate}"'] & F(u)+C \arrow[u, "\text{substitute back}"']
\end{tikzcd}$$

Named for Pierre-Simon Laplace (1749-1827)

- French mathematician
- came from a farming fam. no educated family
- Laplace tutored rich teens to pay for school.
- Two big fields included mathematical astronomy and probability as well as D.E.
- Elected to the French academy @ 23.
- survived the French rev.
- arrogant ... politically "flexible."

Dfn. The Laplace Transform

Let f be a fct. defined for $t \geq 0$.
Then the integral
$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$
is the Laplace Transform of f, provided the integral converges.

Notice $\mathcal{L}\{f(t)\}$ is a fct of $\underline{\underline{s}}$.

Notation: $\mathcal{L}\{f(t)\} = F(s)$
$\mathcal{L}\{g(t)\} = G(s)$
$\mathcal{L}\{y(t)\} = Y(s)$

ex1: $\mathcal{L}\{1\} = \frac{1}{s}$

ex2: $\mathcal{L}\{t^2\} = \frac{2}{s^3}$

ex3: $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

ex4: $\mathcal{L}\{\cos kt\} = \frac{s}{s^2+k^2}$

Important: $\mathcal{L}$ is linear.
$$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha \mathcal{L}\{f(t)\} + \beta \mathcal{L}\{g(t)\} = \alpha F(s) + \beta G(s)$$

ex5: $\mathcal{L}\{f(t)\}$ where $f(t) = \begin{cases} 0 & , 0 \le t < \pi/2 \\ \cos t & , t \ge \pi/2 \end{cases}$

$$\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$$

$$= \int_0^{\pi/2} e^{-st} \cdot 0 \, dt + \int_{\frac{\pi}{2}}^\infty e^{-st} \cos t \, dt$$

$$= 0 + \frac{e^{-st}}{s^2+1}(-s\cos t + \sin t)\Big|_{\frac{\pi}{2}}^\infty$$

$$= \frac{e^{-\frac{\pi}{2}s}}{s^2+1}\Big(\underbrace{-\sin\frac{\pi}{2}}_{1} + s\underbrace{\cos\frac{\pi}{2}}_{0}\Big)$$

$$= -\frac{e^{-\pi/2\, s}}{s^2+1}$$

so when does the Laplace transform exist?

<u>Thm</u>: Sufficient conditions for the existence of $\mathcal{L}$.

If f is piecewise cont. on $[0,\infty)$ and of exponential order, then $\mathcal{L}\{f(t)\}$ exists for $s>c$.

A fct. f is <u>piecewise continuous</u> on $[0,\infty)$ if in any interval $0 \leq a \leq t \leq b$ there are at most a finite number of points t_k, $k=1,2,\dots,n$ w/ $(t_{k-1} < t_k)$ at which f has finite discontinuities and is ~~c~~ cont on each open interval (t_{k-1}, t_k).

A fct f is said to be of <u>exp. order</u> if $\exists$ $c, M > 0$, and $T > 0$ s.t. $|f(t)| \leq Me^{ct}$ for all $t > T$.