6.2: Solutions about ordinary points

Solve 2nd order DE w/ power series.

ex1: $(x^2-4)y'' + 3xy' + y = 0$

$\rightarrow \quad y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$

$y' = c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \dots$

$\rightarrow$ and $3xy' = \quad 3c_1 x + 3\cdot 2 c_2 x^2 + 3\cdot 3 c_3 x^3 + \dots$

$y'' = 1\cdot 2 c_2 + 2\cdot 3 c_3 x + 3\cdot 4 c_4 x^2$

$\rightarrow \quad -4y'' = -4\cdot 1\cdot 2 c_2 - 4\cdot 2\cdot 3 c_3 x - 4\cdot 3\cdot 4 c_4 x^2 - 4\cdot 4\cdot 5 c_5 x^3$

$\rightarrow \quad x^2 y'' = \quad 1\cdot 2 c_2 x^2 + 2\cdot 3 c_3 x^3 + 3\cdot 4 c_4 x^4$

ADD these.

so $(x^2-4)y'' + 3xy' + y = c_0 + -8c_2 + (c_1 + 3c_1 - 24c_3)x +$

$$\sum_{k=2}^{\infty} \left[c_k + 3k c_k - 4(k+1)(k+2)c_{k+2} + (k-1)k\, c_k \right] x^k$$

$c_2 = \frac{1}{8} c_0$ and $c_3 = \frac{1}{6} c_1$

and $(k^2 + 2k + 1)c_k - 4(k+1)(k+2)c_{k+2} = 0$

$$\Rightarrow \quad c_{k+2} = \frac{(k+1)^{\cancel{2}} c_k}{+4\cancel{(k+1)}(k+2)} = \frac{k+1}{+4(k+2)} c_k$$

recurrence relation.

Find the coefficients.

c_0

c_1

$c_2 = \frac{1}{8} c_0$

$c_3 = \frac{1}{6} c_1$

$k=2 \quad c_4 = \frac{3}{+16} c_2 = + \frac{3}{128} c_0$

$c_5 = \frac{4}{+4(5)} c_3 = + \frac{1}{30} c_1$

$k=4 \quad c_6 = \frac{5}{+24} c_4 = \frac{5}{1024} c_0$

$c_6 = \frac{6}{+28} c_5 = \frac{1}{140} c_1$

so $\quad y_1 = c_0\left(1 + \frac{1}{8}x^2 + \frac{3}{128}x^4 + \frac{5}{1024}x^6 + \dots\right)$

$y_2 = c_1\left(x + \frac{1}{6}x^3 + \frac{1}{30}x^5 + \frac{1}{140}x^7 + \dots\right)$

How do we know if/when a power series soln to a linear DE exists?

If a linear DE is in <u>standard form</u> and the coefficients are all defined @ $x = x_0$ then we say x_0 is an <u>ordinary pt.</u> otherwise we call it a singular point.

Thm: If $x = x_0$ is an ordinary pt of $y'' + P(x)y' + Q(x)y = 0$, we can always find two L.I. soln. in the form of a power series centered @ x_0.

$$y = \sum_{n=0}^{\infty} c_n (x - x_0)^n$$

A power series converges at least on some interval $|x - x_0| < R$ where R is the dist. from x_0 to the closest singular pt.

The soln to

ex1 rev: $(x^2-4)y'' + 3xy' + y = 0$ converges at least for $-2 < x < 2$.

ex2: Solve $y'' - xy' - x^2 y = 0$

$$y = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$$

$$x^2 y = c_0 x^2 + c_1 x^3 + \cdots$$

$$y' = 1 \cdot c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \cdots$$

$$x y' = 1c_1 x + 2c_2 x^2 + 3c_3 x^3 + \cdots$$

$$y'' = 1 \cdot 2 c_2 + 2 \cdot 3 c_3 x + 3 \cdot 4 c_4 x^2 + 4 \cdot 5 c_5 x^3 + \cdots$$

so $0 = y'' - xy' - x^2 y = 2c_2 + (6c_3 - c_1)x + \sum_{k=2}^{\infty} \left[(k+1)(k+2)c_{k+2} - kc_k - c_{k-2} \right] x^k$

$\Rightarrow \quad c_2 = 0 \quad$ and $\quad c_3 = \frac{1}{6} c_1$.

the recurrence relation is $c_{k+2} = \dfrac{c_{k-2} + k c_k}{(k+1)(k+2)}$

c_0

c_1

$c_2 = 0$

$c_3 = \frac{1}{6} c_1$

$k=2 \qquad c_4 = \dfrac{c_0 + 2c_2}{3 \cdot 4}$

$c_5 = \dfrac{c_1 + 3c_3}{4 \cdot 5}$

$c_6 = \dfrac{c_2 + 4c_4}{5 \cdot 6}$

The simplest way to work w/a 3 term recurrence relation is to consider cases,

<u>case1:</u> $c_0 \neq 0, \; c_1 = 0$

c_0

$c_1 = 0$

$c_2 = 0$

$c_3 = 0$

$c_4 = \frac{1}{12} c_0$

$c_5 = 0$

$c_6 = \frac{1}{90} c_0$

$c_7 = 0$

$c_8 = \dfrac{c_4 + 6c_6}{7 \cdot 8} = \dfrac{3}{1120} c_0$

case 2: $c_0 = 0, \; c_1 \neq 0$

$c_0 = 0$

c_1

$c_2 = 0$

$c_3 = \frac{1}{6} c_1$

$c_4 = 0$

$c_5 = \frac{3}{40} c_1$

$c_6 = 0$

$c_7 = \dfrac{c_3 + 5c_5}{6 \cdot 7} = \dfrac{13}{1008}$

So the two L.I. solns are:

$$y_1 = c_0\left(1 + \frac{1}{12}x^4 + \frac{1}{90}x^6 + \frac{3}{1120}x^8 + \dots\right)$$

$$y_2 = c_1\left(x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \frac{13}{1008}x^7 + \dots\right)$$