6.1: Review of Power Series

Form: $\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \ldots$

power series centered @ $x=a$.

ex: $\sum_{n=0}^{\infty} (-1)^n \frac{(x-2)^n}{n!}$

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \ldots$$

ex: $\sum_{n=0}^{\infty} \frac{x^n}{n!}$

Vocab:

Convergence: $\sum_{n=0}^{\infty} c_n (x-a)^n$ is convergent at x if $\lim_{N\to\infty} \sum_{n=0}^{N} c_n (x-a)^n$ exists.

interval of convergence

radius of convergence

absolute convergence

Test for convergence: Ratio Test

$\sum_{n=0}^{\infty} a_n$ converges absolutely if

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| < 1$$

if the limit $= 0$, the test is inconclusive,

limit > 0, the series diverges.

ex1! Find the I.O.C. of $\sum_{k=1}^{\infty} \frac{1}{k^2+k}(3x-1)^k$

$$\lim_{k\to\infty} \left| \frac{(k^2+k)(3x-1)^{k+1}}{((k+1)^2+(k+1))(3x-1)^k} \right| = |3x-1| \lim_{k\to\infty} \left| \frac{k^2+k}{k^2+3k+2} \right|$$

$$= |3x-1|$$

Abs. converges when $-1 < 3x-1 < 1$

$0 < 3x < 2$

$0 < x < \frac{2}{3}$

It will converge conditionally @ the endpts by the LCT w/ $\sum \frac{1}{k^2}$

So I.O.C. is $[0, \frac{2}{3}]$

and R.O.C. is $\frac{1}{3}$

In the next section, we will need to combine series which requires all terms being written w/ x^k.

ex 2: $\sum_{n=2}^{\infty}(n-1)n c_n x^{n-2} - 2\sum_{n=1}^{\infty} n c_n x^n$

$= 1\cdot2 c_2 x^0 + 2\cdot3 c_3 x^1 + 3\cdot4 c_4 x^2 + \dots$
$\quad - 2\cdot1 c_1 x^1 - 2\cdot2 c_2 x^2 - 2\cdot3 c_3 x^3 - \dots$

$= 2c_2 + \sum_{k=1}^{\infty}\left[(k+1)(k+2)c_{k+2} - 2k c_k\right]x^k$

We can us a power series of the form $y = \sum_{n=0}^{\infty} c_n x^n$ to solve DE.

ex3: $y' = xy \Rightarrow y' - xy = 0$

Let $y = c_0 + c_1 x + c_2 x^2 + \dots$

$\Rightarrow xy = c_0 x + c_1 x^2 + c_2 x^3 + \dots$

and $y' = 1c_1 + 2c_2 x + 3c_3 x^2 + 4c_4 x^3 + \dots$

$0 = y' - xy = c_1 + \sum_{k=1}^{\infty}\left[(k+1)c_{k+1} - c_{k-1}\right]x^k$

$\Rightarrow c_1 = 0$ and $c_{k+1} = \frac{c_{k-1}}{k+1}$

$\swarrow$

↳ c_0

$c_1 = 0$

k=1 $\quad c_2 = \frac{c_0}{2}$

$c_3 = 0$

k=3 $\quad c_4 = \frac{c_2}{4} = \frac{c_0}{8}$

k=4 $\quad c_5 = 0$

$c_6 = \frac{c_4}{6} = \frac{c_0}{48}$

so $\quad y = c_0\left(1 + \frac{1}{2}x^2 + \frac{1}{2\cdot 4}x^4 + \frac{1}{2\cdot 4\cdot 6}x^6 + \dots\right)$

$$= c_0 \sum_{k=0}^{\infty} \frac{1}{k!}\left(\frac{x^3}{2}\right)^k$$