4.7: Cauchy - Euler Equation

Form: $a_n x^n y^{(n)} + \ldots + a_2 x^2 y'' + a_1 x y' + a_0 y = 0$

$n=3$ $\quad a_3 x^3 y''' + a_2 x^2 y'' + a_1 x y' + a_0 y = 0$

$n=2$ $\quad a_2 x^2 y'' + a_1 x y' + a_0 y = 0$

Soln. of the form $y = x^m$ and look for m.

$n=2$: $\quad a_2 x^2 \cdot m(m-1) x^{m-2} + a_1 x \cdot m x^{m-1} + a_0 x^m = 0$

$\Rightarrow x^m \left(a_2 m(m-1) + a_1 m + a_0 \right) = 0$

on $x > 0$, $\quad a_2 m(m-1) + a_1 m + a_0 = 0$

⋮

solve for m.

Also called an equidimensional eqt.

If y in meters

x in seconds

y, xy', $x^2 y''$ have the same units

ex1: $3x^2 y'' + 11 x y' - 3y = 0$

$\Rightarrow 3m(m-1) + 11m - 3 = 0$

$\Rightarrow 3m^2 + 8m - 3 = 0$

$\Rightarrow (3m-1)(m+3) = 0$

$\Rightarrow m = \frac{1}{3}$ or $m = -3$

so $y = c_1 x^{1/3} + c_2 x^{-3}$

ex 2: $9t^2 y'' + 15ty' + y = 0$

$\Rightarrow 9m(m-1) + 15m + 1 = 0$

$\Rightarrow 9m^2 + 6m + 1 = 0$

$\Rightarrow (3m+1)^2 = 0$

$\Rightarrow m = -\frac{1}{3}$ (mult. 2).

so $y_1 = t^{-1/3}$

Then use section 4.2 to find the second solution $y_2 = t^{-1/3} \ln t$

ex3: $x^2y'' + 5xy' + 5y = 0$

$\Rightarrow m(m-1) + 5m + 5 = 0$

$\Rightarrow m^2 + 4m + 5 = 0$

$\Rightarrow m = \dfrac{-4 \pm \sqrt{16 - 4(1)(5)}}{2}$

$= -2 \pm i$

$\Rightarrow y_1 = x^{-2+i}$ and $y_2 = x^{-2-i}$

recall: $e^{\ln x} = x$

$x^{-2+i} = x^{-2}x^{i}$

$= x^{-2}(e^{\ln x})^{i}$

$= x^{-2}e^{i\ln x}$

$= x^{-2}(\cos(\ln x) + i\sin(\ln x))$

as before, we can find a real fundamental set $y_1 = x^{-2}\cos(\ln x)$ and $y_2 = x^{-2}\sin(\ln x)$

ex 4: $x^2y'' + 10xy' + 8y = x^2$

$\Rightarrow m(m-1) + 10m + 8 = 0$

$\Rightarrow m^2 + 9m + 8 = 0$

$\Rightarrow (m+8)(m+1) = 0$

$\Rightarrow m = -8$ or $m = -1$

$\Rightarrow y_c = c_1 x^{-1} + c_2 x^{-8}$

Find the particular soln. w/variation of parameters

$$W = \begin{vmatrix} x^{-1} & x^{-8} \\ -x^{-2} & -8x^{-9} \end{vmatrix} = -7x^{-10}$$

Must start by putting the DE in standard form

$$W_1 = \begin{vmatrix} 0 & x^{-8} \\ x^2 & -8x^{-9} \end{vmatrix} = -x^{-6}$$

$$W_2 = \begin{vmatrix} x^{-1} & 0 \\ -x^{-2} & x^2 \end{vmatrix} = x$$

$u_1' = \frac{1}{7}x^4 \Rightarrow u_1 = \frac{1}{35}x^5$

$u_2' = -\frac{1}{7}x^{11} \Rightarrow u_2 = -\frac{1}{84}x^{12}$

$y_p = \frac{1}{35}x^5 \cdot x^{-1} + -\frac{1}{84}x^{12} \cdot x^{-8}$

$= \frac{1}{60}x^4$

$\frac{1}{30}x^2$

$y = c_1 x^{-1} + c_2 x^{-8} + \frac{1}{60}x^4$