

4.6: Variation of Parameters

Remember how when we reduced the order, we started w/ a known soln and let $y = u^*$ (known soln). Then we found u and worked our way back to the soln.

We will focus on 2nd order in standard form.

$$y'' + P(x)y' + Q(x)y = f(x)$$

Start finding the solns y_1, y_2 to the homogeneous eq. $y_2 = c_1 y_1 + c_2 y_2$

We guess $y_p = u_1 y_1 + u_2 y_2$

To find u_1, u_2

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \quad W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}, \quad \& \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

and $u_1' = \frac{W_1}{W}$ and $u_2' = \frac{W_2}{W}$

ex 1: $y'' + y = \sin x$
 $y = c_1 \cos x + c_2 \sin x - \frac{1}{2} x \cos x$

ex 2: $y'' - y = \cosh x$
 $y = c_3 e^x + c_4 e^{-x} + \frac{1}{2} x \sinh x$

ex 3: $y'' + 3y' + 2y = \frac{1}{1+e^x}$
 $y = c_3 e^{-x} + c_2 e^{-2x} + (1+e^x)e^{-x} \ln(1+e^x)$

ex 4: $y'' + 2y' + y = e^{-x} \ln x$
 $y = c_1 e^{-x} + c_2 x e^{-x} + \frac{1}{2} x^2 e^{-x} \ln x - \frac{3}{4} x^2 e^{-x}$

Derivation of: Variation of Parameters

Note: You can solve a system of linear equations w/ determinants.

$$\begin{cases} ax + by = c \\ dx + ey = f \end{cases} \rightarrow x = \frac{\omega_1}{\omega} \quad \text{and} \quad y = \frac{\omega_2}{\omega}$$

where $\omega = \begin{vmatrix} a & b \\ d & e \end{vmatrix}$; $\omega_1 = \begin{vmatrix} c & b \\ f & e \end{vmatrix}$; $\omega_2 = \begin{vmatrix} a & c \\ d & f \end{vmatrix}$

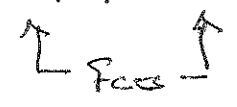
ex: solve $\begin{cases} 3x + y = 1 \\ x - y = 7 \end{cases}$

The 2nd order DE: $a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$

$$\Rightarrow y'' + P(x)y' + Q(x)y = f(x)$$

We can find a solution to the associated homogeneous DE: $y_h = c_1 y_1(x) + c_2 y_2(x)$.

Just like when we reduced the order, we write our particular soln. as: $y_p = u_1 y_1 + u_2 y_2$



$$y_p' = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2'$$

$$y_p'' = u_1'' y_1 + u_1' y_1' + u_1 y_1'' + u_2'' y_2 + u_2' y_2' + u_2 y_2'' + u_2' y_2' + u_2 y_2''$$

$$\begin{aligned} \Rightarrow y'' + P y' + Q y &= u_1'' y_1 + u_1' y_1' + u_1 y_1'' + u_2'' y_2 + u_2' y_2' + u_2 y_2'' + \\ &u_2' y_2' + u_2 y_2'' + P u_1' y_1 + P u_1 y_1' + P u_2' y_2 + \\ &P u_2 y_2' + Q u_1 y_1 + Q u_2 y_2 \end{aligned}$$

$$\begin{aligned}
 &= u_1(0) + u_2(0) + u_1'' y_1' + u_1' y_1'' + \\
 &\quad u_1' y_1' + u_2'' y_2' + u_2' y_2'' + u_2' y_2' + P u_1' y_1' + \\
 &\quad P u_2' y_2' \\
 &= \frac{d}{dx} [u_1' y_1'] + \frac{d}{dx} [u_2' y_2'] + P [u_1' y_1' + u_2' y_2'] + \\
 &\quad u_1' y_1'' + u_2' y_2'' \\
 &= \frac{d}{dx} [u_1' y_1' + u_2' y_2'] + P [u_1' y_1' + u_2' y_2'] + u_1' y_1'' + u_2' y_2'' \\
 &= f(x)
 \end{aligned}$$

Since we need 2 eqs to go w/ the two unknowns u_1 and u_2 , assume $u_1' y_1' + u_2' y_2' = 0$

$\Rightarrow u_1' y_1' + u_2' y_2' = f(x)$, so w/ variables u_1', u_2'

$$\begin{cases} y_1 u_1' + y_2 u_2' = 0 \\ y_1' u_1' + y_2' u_2' = f(x) \end{cases} \Rightarrow u_1' = \frac{\omega_1}{\omega} \text{ and } u_2' = \frac{\omega_2}{\omega}$$

$$\omega = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}; \quad \omega_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}; \quad \omega_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$