4.6: Variation of Parameters

Remember how when we reduced the order, we started w/ a known soln and let $y = u^*(\text{known soln})$. Then we found u and worked our way back to the soln.

We will focus on 2^{nd} order in standard form.

$$y'' + P(x)y' + Q(x)y = f(x)$$

Start finding the solns y_1, y_2 to the homogeneous eqt. $y_c = c_1 y_1 + c_2 y_2$

we guess $y_p = u_1 y_1 + u_2 y_2$

To find u_1, u_2

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}, \quad W_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix}, \quad \& \quad W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$

and $u_1' = \dfrac{W_1}{W}$ and $u_2' = \dfrac{W_2}{W}$.

ex 1: $y'' + y = \sin x$

$$y = c_1 \cos x + c_2 \sin x - \tfrac{1}{2} x \cos x$$

ex 2: $y'' - y = \cosh x$

$$y = c_3 e^{x} + c_4 e^{-x} + \tfrac{1}{2} x \sinh x$$

ex 3: $y'' + 3y' + 2y = \dfrac{1}{1+e^{x}}$

$$y = c_1 e^{-x} + c_2 e^{-2x} + (1+e^{x})e^{-x} \ln(1+e^{x})$$

ex 4: $y'' + 2y' + y = e^{-t} \ln t$

$$y = c_1 e^{-t} + c_2 t e^{-t} + \tfrac{1}{2} t^2 e^{-t} \ln t - \tfrac{3}{4} t^2 e^{-t}$$

Derivation of: Variation of Parameters

Note: You can solve a system of linear equations w/ determinants.

$$\begin{cases} ax + by = c \\ dx + ey = f \end{cases} \longrightarrow x = \frac{W_1}{W} \quad \text{and} \quad y = \frac{W_2}{W}$$

where $W = \begin{vmatrix} a & b \\ d & e \end{vmatrix}$; $W_1 = \begin{vmatrix} c & b \\ f & e \end{vmatrix}$; $W_2 = \begin{vmatrix} a & c \\ d & f \end{vmatrix}$

ex: solve $\begin{cases} 3x + y = 1 \\ x - y = 7 \end{cases}$

The 2nd order DE: $a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$

$\Rightarrow y'' + P(x)y' + Q(x)y = f(x)$.

We can find a solution to the associated homogeneous DE: $y_c = c_1 y_1(x) + c_2 y_2(x)$.

Just like when we reduced the order, we write our particular soln. as: $y_p = u_1 y_1 + u_2 y_2$

fcts

$$y_p' = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2'$$

$$y_p'' = u_1'' y_1 + u_1' y_1' + u_1' y_1' + u_1 y_1'' + u_2'' y_2 + u_2' y_2' + u_2' y_2' + u_2 y_2''$$

$$\Rightarrow y'' + Py' + Qy = u_1'' y_1 + u_1' y_1' + u_1' y_1' + u_1 y_1'' + u_2'' y_2 + u_2' y_2' + u_2' y_2' + u_2 y_2'' + Pu_1' y_1 + Pu_1 y_1' + Pu_2' y_2 + Pu_2 y_2' + Qu_1 y_1 + Qu_2 y_2$$

$$= u_1(0) + u_2(0) + u_1'y_1 + u_1'y_1' + u_1'y_1' + u_2''y_2 + u_2'y_2' + u_2'y_2' + P u_1'y_1 + P u_2'y_2$$

$$= \frac{d}{dx}[u_1'y_1] + \frac{d}{dx}[u_2'y_2] + P[u_1'y_1 + u_2'y_2] + u_1'y_1' + u_2'y_2'$$

$$= \frac{d}{dx}[u_1'y_1 + u_2'y_2] + P[u_1'y_1 + u_2'y_2] + u_1'y_1' + u_2'y_2'$$

$$= f(x).$$

since we need 2 eqns to go w/ the two unknown fcts u_1 and u_2, assume $u_1'y_1 + u_2'y_2 = 0$

$\Rightarrow u_1'y_1' + u_2'y_2' = f(x)$. so w/ variables u_1', u_2'

$$\begin{cases} y_1u_1' + y_2u_2' = 0 \\ y_1'u_1' + y_2'u_2' = f(x) \end{cases} \Rightarrow u_1' = \frac{w_1}{w} \text{ and } u_2' = \frac{w_2}{w}$$

$$w = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} ; \quad w_1 = \begin{vmatrix} 0 & y_2 \\ f(x) & y_2' \end{vmatrix} ; \quad w_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & f(x) \end{vmatrix}$$