4.3: Homogeneous Linear Eqts w/ const coef.

<u>ex 1</u>: Let's start w/ a simple example $3y' - 7y = 0$

$\Rightarrow \quad y' = \frac{7}{3} y$

we know the soln: $y = e^{\frac{7}{3}x}$

Looking ahead, this soln. is of the form $y = e^{mx}$

so $3y' - 7y = 0 \Rightarrow 3me^{mx} - 7e^{mx} = 0$

$$\Rightarrow \underbrace{e^{mx}}_{\neq 0} \underbrace{(3m - 7)}_{m = 7/3} = 0$$

Too cool... we can solve a DE w/ a good guess and a little algebra.

<u>ex 2</u>: $3y'' + 13y' - 10y = 0$

could our solns. (again) be of the form $y = e^{m}$?

$\Rightarrow 3m^2 e^{mx} + 13me^{mx} - 10e^{m} = 0$

$\Rightarrow e^{mx}(3m^2 + 13m - 10) = 0$

$\Rightarrow (3m - 2)(m + 5) = 0$

$\Rightarrow \quad m = \frac{2}{3}$ and $m = -5$

so our gen. sol is $y = c_1 e^{\frac{2}{3}x} + c_2 e^{-5x}$.

Wow, this is easy!

<u>ex 3</u>: $10y'' + 11y' + 3y = 0$.

If the soln. is like $y = e^{mx}$, then $10m^2 + 11m + 3 = 0$

$\Rightarrow (2m + 1)(5m + 3) = 0$

and so $y = c_1 e^{-\frac{1}{2}x} + c_2 e^{-\frac{3}{5}x}$.

This is boring... so much so you may walk out.

ex 4: $y'' - 2y' + y = 0$

$\Rightarrow \quad m^2 - 2m + 1 = 0$

$\Rightarrow \quad (m-1)^2 = 0$

$\Rightarrow \quad m = 1$ w/ multiplicity 2.

so $y = c_1 e^x + c_2 e^x$?

$= c_3 e^x$?

These aren't L.I. soln., but we know there should be 2 such soln.

Since we have one... use reduction of order.

$y_1 = e^x$

$$y_2 = e^x \int \frac{e^{-\int -2\,dx}}{e^{2x}}\,dx$$

$$= e^x \int \frac{e^{2x}}{e^{2x}}\,dx$$

$$= x e^x$$

so $y = c_1 e^x + c_2 x e^x$.

ex 5: $y^{(4)} - 4y^{(3)} + 6y'' - 4y' + y = 0$

$\Rightarrow m^4 - 4m^3 + 6m^2 - 4m + 1 = 0$

$\Rightarrow (m-1)^4 = 0$

and $y = c_1 e^x + c_2 x e^x + c_3 x^2 e^x + c_4 x^3 e^x$.

clearly polynomial issues are in play. so what about complex soln.

ex 6: $y'' - 4y' + 13y = 0$

$$\Rightarrow \quad m^2 - 4m + 13 = 0$$

$$\Rightarrow \quad m = \frac{4 \pm \sqrt{16 - 4(1)(13)}}{2}$$

$$= 2 \pm 3i$$

$$\Rightarrow \quad y = c_1 e^{(2+3i)x} + c_2 e^{(2-3i)x}$$

$$= c_1 e^{2x} e^{3ix} + c_2 e^{2x} e^{-3ix}$$

$$= e^{2x}\left(c_1 e^{3ix} + c_2 e^{-3ix}\right)$$

recall: $e^{i\theta} = \cos\theta + i\sin\theta$

so $e^{3ix} = \cos 3x + i \sin 3x$

and $e^{-3ix} = \cos 3x - i \sin 3x$

If $e^{2x}e^{3ix}$ and $e^{2x}e^{-3ix}$ are a fundamental set, then

So are $e^{2x}(e^{3ix} + e^{-3xi}) = e^{2x} \cdot 2\cos 3x$

and

$e^{2x}(e^{3ix} - e^{-3xi}) = e^{2x} \cdot 2i \sin 3x$

or even better $e^{2x}\cos 3x$ and $e^{2x}\sin 3x$

Hence, the general soln is $y = e^{2x}(c_1 \cos 3x + c_2 \sin 3x)$.

ex 7: $y'' - 10y' + 41y = 0$

$$\Rightarrow \quad m^2 - 10m - 41 = 0$$

$$\Rightarrow \quad m = 5 \pm 4i$$

$$\Rightarrow \quad y = e^{5x}(c_1 \cos 4x + c_2 \sin 4x)$$