4.2 : Reduction of order

Basic idea: Use a known solution to find a second solution.

$y_1(x) \leftarrow$ known

$y_2(x) \leftarrow$ unknown

Find y_2 by first finding u s.t. $y_2 = u \cdot y_1$

We will focus entirely on 2^{nd} order examples

ex: solve $x^2y'' - 5xy' + 9y = 0$ given that $y_1 = x^3$ is a soln.

In standard form: $y'' - \frac{5}{x}y' + \frac{9}{x^2}y = 0$

$$y_2 = x^3 \cdot u$$
$$y_2' = 3x^2u + x^3u'$$
$$y_2'' = 6xu + 3x^2u' + 3x^2u' + x^3u''$$
$$= 6xu + 6x^2u' + x^3u''$$

So $(6xu + 6x^2u' + x^3u'') - \frac{5}{x}(3x^2u + x^3u') + \frac{9}{x^2}x^3u = 0$

$\Rightarrow u(6x - 15x + 9x) + u'(6x^2 - 5x^2) + u'' \cdot x^3 = 0$

$\Rightarrow u'' + \frac{1}{x}u' = 0$ if $u' = w$

$\Rightarrow w' + \frac{1}{x}w = 0$ $\quad P(x) = \frac{1}{x}$ $\quad \int P(x)dx = \ln|x|$ $\quad e^{\int P(x)dx} = x$

$\Rightarrow xw' + w = 0$

$\Rightarrow \frac{d}{dx}(xw) = 0$

$\Rightarrow \quad x \cdot w = c$

$\Rightarrow \quad w = \frac{c}{x}$

$\Rightarrow \quad u' = \frac{c}{x}$

$\Rightarrow \quad u = c\ln|x| + d.$

and $y_2 = x^3(c\ln x + d)$ or $y_2 = x^3 \ln x$

w/ general soln: $y = c_1 x^3 + c_2 x^3 \ln x$

$$y_2 = y_1 \int \frac{e^{-\int P(x)\,dx}}{y_1^2}\,dx$$

Formula

The process

(1) write in standard form
$y'' + P(x)y' + Q(x)y = 0$

(2) Given y_1, let $y = u y_1$

(3) find y' and y'' and sub into (1). Group by the order of u.

(4) reduce the order w/ $w = u'$ and solve the linear eqt. for w.

(5) sub back and solve for u

(6) find $y = u y_1$, a second soln, and the general solution

ex: Given $y_1 = x$, find the gen. soln of
$(x^2 - 1)y'' - 2xy' + 2y = 0, \quad x^2 < 1$

soln: $y = c_1 x + c_2(1 + x^2)$