4.1: Prelim. Theory: Linear Eqts.

Comparing the linear n^{th} order homogeneous and nonhomogeneous DE.

homogeneous: $a_n(x)y^{(n)} + \ldots + a_1(x)y' + a_0(x)y = 0$

Non-homogeneous: $a_n(x)y^{(n)} + \ldots + a_1(x)y' + a_0(x)y = g(x)$

There are two types of solutions: general and particular solutions. A general soln is a family of fcns that satisfy the DE. A particular solution doesn't have unknown constants.

ex: DE $y'' - y = 0$

General sol: $y = c_1e^{x} + c_2e^{-x}$

particular soln: $y = 2e^{x} + 3e^{-x}$

Is a particular soln unique? Obviously this would require more info

The IVP

$$a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \ldots + a_1(x)y' + a_0(x)y = g(x)$$

subject to: $y(x_0) = y_0,\ y'(x_0) = y_1,\ \ldots,\ y^{(n-1)}(x_0) = y_{n-1}$

key: same x-value.

Thm: Existence of a unique soln.

Let $a_n(x), a_{n-1}(x), \dots, a_1(x), a_0(x)$ and $g(x)$ be cont. on an interval I and let $a_n(x) \neq 0$ on I. If $x_0 \in I$ then a soln $y(x)$ of the IVP exists and is unique

This is analogous to what we did previously.

The BVP (Boundary-Value Problem). These have an order of at least two.

Now we have $a_n(x)y^{(n)} + \dots + a_1(x)y' + a_0(x)y = g(x)$

subject to: $y(x_0) = y_0$, $y(x_1) = y_1$, ..., $y(x_{n-1}) = y_{n-1}$

or: $y(x_0) = y_0$, $y'(x_1) = y_1$, ..., $y^{(n-1)}(x_{n-1}) = y_{n-1}$

key: different (more than one) x-value. But there can be many configs of constraints.

Unlike the IVP, a BVP can have 0, 1, or many solns.

See ex3 for this ... this added complexity is why we mostly focus on IVP's.

The differential operator

$\frac{d}{dx}(y)$	$f'(x)$	Dy	ex: $D(2x^4 - 3x)$
$\frac{dy}{dx}$	y'	$D(y)$	$D^2(2x^4 - 3x)$
$\frac{df}{dx}$	$\dot{y}$	$D(f)$	

ex: $y'' - y = 0$

$\Rightarrow (D^2 - 1)y = 0$

The D notation isn't overly critical, what matters is that the differential operator is <u>linear</u>

<u>Dfn</u>: A fct L is linear iff $L(\alpha x + y) = \alpha L(x) + L(y)$

<u>ex</u>: $L_1(x) = 7x$

$\Rightarrow L(\alpha x + y) = 7(\alpha x + y) = \alpha(7x) + 7y =$

<u>ex</u>: $L_2(x) = 7x + 1$

$\Rightarrow L(\alpha x + y) = 7(\alpha x + y) + 1 = 7\alpha x + 7y + 1$

and $\alpha L(x) + L(y) = \alpha(7x+1) + (7y+1) = 7\alpha x + 7y + 1 + \alpha$

These are not equal, so L_2 is not linear.

More generally, $y = mx + b$, $m, b \neq 0$ is not linear.

<u>ex</u>: $Df = \frac{df}{dx}$

$\Rightarrow D(\alpha f + g) = \alpha Df + Dg$

We like the fact that D is linear because it allows us to combine solns.

<u>Thm</u>: Superposition Principle - Homogeneous Equations

Let $y_1, \dots, y_k$ be solns of the n^{th} order linear homogeneous DE. Then $y = c_1 y_1 + \dots + c_k y_k$ is also a soln.

ex: if $y_1 = e^x$ and $y_2 = e^{-x}$ are soln to
$y'' - y = 0$, then $y_3 = c_1 e^x + c_2 e^{-x}$ is a soln.

Dfn: $f_1, \dots, f_N$ are <u>linearly dependent</u> (LD) if
$\exists$ $c_1, \dots, c_N$ not all zero s.t. $c_1 f_1 + \dots + c_N f_N = 0$

Dfn: If no such c exist, the fcts are
linearly independent (LI).

The idea of LI is analagous to LI in
linear algebra.

You can check for LI w/ the wronskian

Dfn: Suppose $f_1, \dots, f_N$ possess at least $N-1$ derivatives

$$W = \begin{vmatrix} f_1 & f_2 & \cdots & f_N \\ f_1' & f_2' & & f_N' \\ \vdots & \vdots & & \vdots \\ f_1^{(N-1)} & f_2^{(N-1)} & & f_N^{(N-1)} \end{vmatrix}$$ is the wronskian

and $f_1, \dots, f_N$ are L.I when $W \neq 0$.

ex: verify e^{2x}, e^{-3x} are LI on $\mathbb{R}$

Dfn: Any set $y_1, \dots, y_N$ of N LI soln of the
homogeneous linear N^{th} order DE is said to
be a <u>fundamental set</u> of solns on I.

In linear algebra we call this a basis.

The good news is that this fundamental set exists (our search will be fruitful).

The general soln: If $y_1, \dots, y_n$ are a fundamental set, then the general soln. is $y = c_1 y_1 + \dots + c_n y_n$

ex: e^{2x} and e^{-3x} are soln. to $y'' - y' - 6y = 0$.
Find the general soln.

Thm: General Soln - Nonhomogeneous eqts.

Let y_p be a particular soln to the DE
$a_n(x) y^{(n)} + \dots + a_1(x) y' + a_0(x) y = g(x)$ and
let $y_1, \dots, y_n$ be a fundamental set of the associated homogeneous DE. Then the general soln is: $y = c_1 y_1 + \dots + c_n y_n + y_p$

This is due to D being linear.

In $$y = \underbrace{c_1 y_1 + \dots + c_n y_n}_{\text{complementary fct}} + \underbrace{y_p}_{\text{particular soln}}$$

So $y = y_c + y_p$

Thm: Superposition Principle - nonhomogeneous Eqts.

Let $y_{p_1}, \dots, y_{p_k}$ be k particular solutions corresponding to $a_n(x)y^{(n)} + \dots + a_1(x)y' + a_0(x)y = g_i(x)$ for $i = 1, 2, \dots, k$

Then $y_p = y_{p_1} + \dots + y_{p_k}$ is a soln to

$$a_n(x)y^{(n)} + \dots + a_1(x)y' + a_0(x)y = g_1(x) + \dots + g_k(x)$$

Yeah for linearity!

This is gonna be handy latter on when $g(x)$ has multiple terms.