3.1: Linear Models

ex1: A pop. of bacteria grows at a rate proportional to the size of the population. If the pop increases 6 fold in 10hr, then what is the doubling time?

t = time in hrs

$P(t)$ = pop @ time t.

$P(0) = P_0$

$$\frac{dP}{dt} = kP$$

$$\Rightarrow \frac{dP}{P} = k\,dt$$

$$\Rightarrow \ln|P| = kt + c$$

$$\Rightarrow P = P_0 e^{kt}$$

Solve for k

$$6P_0 = P_0 e^{10k}$$

$$\Rightarrow k = \frac{\ln(6)}{10} = 0.1792$$

and the doubling time

$$2P_0 = P_0 e^{kt}$$

$$\Rightarrow 2 = e^{kt}$$

$$\Rightarrow t = \frac{\ln 2}{k} = 3.8685$$

The pop. doubles in just under 4 hours.

ex2: The rate at which ^{14}C decays in a dead (but once living) object is proportional the amount of ^{14}C present. Experimentally we have found the $\frac{1}{2}$ life of ^{14}C to be 5730 years.

Carbon taken from a purported relic of the time of Christ contained 4.6×10^{10} atoms of ^{14}C per gram. Carbon extracted from a present day specimen of the same substance contained 5.0×10^{10} atoms of ^{14}C per gram. Compute the approx age of the relic.

Let t = # of yrs since an object died.

$A(t)$ = Amt of ^{14}C present @ time t.

so $\frac{dA}{dt} = kA$

$\Rightarrow \frac{dA}{A} = k\,dt$

$\Rightarrow \ln|A| = kt + c$

$\Rightarrow A = A_0 e^{kt}$

Find k

$\frac{1}{2} = e^{k \cdot 5730}$

$\Rightarrow k = \frac{\ln(1/2)}{5730}$

$= -1.2097 \times 10^{-4}$

check the age.

$4.6 \times 10^{10} = 5.0 \times 10^{10} e^{-1.2097 \times 10^{-4} t}$

$\Rightarrow t = \frac{\ln\left(\frac{4.6}{5.0}\right)}{k}$

$= 689$ years

It is unlikely that the relic is authentic.

ex 3: The rate at which an objects temp. changes is proportional to the difference in temp between the object & ambient temp.

A cake is removed from an oven @ 210°F and left to cool @ room temp, which is 70°F. After 30 min the cake is 140°F. When will it be 100°F.

t = time in min

H_r = ambient temp.

H = temp of the object

$$\frac{dH}{dt} = k(H_r - H)$$

$$\Rightarrow \frac{dH}{H_r - H} = k\,dt$$

$$\Rightarrow \ln|H_r - H| = kt + c$$

$$\Rightarrow H_r - H = ce^{kt}$$

$$\Rightarrow H = H_r - ce^{kt}$$

The initial difference

$$c = H_r - H(0)$$

$$= 70 - 210$$

$$= -140$$

Find k

$$140 = 70 + 140e^{30k}$$

$$\Rightarrow \frac{1}{2} = e^{30k}$$

$$\Rightarrow k = \frac{\ln(1/2)}{30} = -0.0231$$

Now find when $H = 100$

$$100 = 70 + 140e^{kt}$$

$$\Rightarrow t = \frac{\ln \frac{3}{14}}{k} = 66.67$$

so it will take a little over an hour for the cake to cool.

ex4: A large tank is filled w/ 500 gal. of pure water. Brine containing 2 lbs of salt/gal is pumped in at 5 gal/min. The well-mixed soln. is pumped out at 10 gal/min. Find the # of lbs of salt in the tank at time t.

t = # of min elapsed since we began pumping the brine in.

A(t) = amt of salt in the tank @ time t.

$A_0 = 0$

$$\frac{dA}{dt} = \left(\underset{in}{\text{rate}}\right) - \left(\underset{out}{\text{rate}}\right)$$

$$= 10 \frac{\text{lbs}}{\text{min}} - \frac{A}{500 - 5t} \cdot 10$$

$$= 10 - \frac{2A}{100 - t}$$

$$P(t) = \frac{2}{100 - t}$$

$$\int P(t)\,dt = -2\ln|100 - t|$$

Integrating factor

$$e^{\int P(t)dt} = \frac{1}{(100-t)^2}$$

$$\Rightarrow A' + \frac{2}{100 - t} A = 10 \quad \text{(linear)}$$

$$\Rightarrow \frac{1}{(100-t)^2} A' + \frac{2}{(100-t)^3} A = \frac{10}{(100-t)^2}$$

$$\Rightarrow \frac{d}{dt}\left[\frac{1}{(100-t)^2} \cdot A\right] = \frac{10}{(100-t)^2}$$

$$\Rightarrow \frac{1}{(100-t)^2} A = \frac{10}{100 - t} + c$$

$$\Rightarrow A = 10(100 - t) + c(100 - t)^2$$

Solve for c

$0 = 1000 + c(10000)$

$\Rightarrow c = -\frac{1}{10}$

$$A(t) = 10(100 - t) - \frac{1}{10}(100 - t)^2$$

3.1: Linear Models

1.) A population of bacteria grows at a rate proportional to the size of the population. If the population increases six fold in 10 hours, then what is its doubling time?

t = time (hrs)

$P(t)$ = pop @ time t.

$$\frac{dP}{dt} = kP$$

$$\Rightarrow \frac{dP}{P} = k\,dt$$

$$\Rightarrow \ln|P| = kt + c$$

$$\Rightarrow P = P_0 e^{kt}$$

~~Sol~~

The pop. doubles in about 4 hours.

solve for k.

$$6P_0 = P_0 e^{k(10)}$$

$$\Rightarrow 6 = e^{k(10)}$$

$$\Rightarrow k = \frac{\ln 6}{10}$$

$$= 0.1792$$

find the doubling time.

$$2P_0 = P_0 e^{kt}$$

$$\Rightarrow 2 = e^{kt}$$

$$\Rightarrow t = \frac{\ln 2}{k}$$

$$= 3.8685$$

2.) The rate at which ^{14}C decays in a dead (but once living) object is proportional to the amount of the ^{14}C present. Experimentally we have found the half-life of ^{14}C to be 5730 years.

Carbon taken from a purported relic of the time of Christ contained $4.6x10^{10}$ atoms of ^{14}C per gram. Carbon extracted from a present day specimen of the same substance contained $5.0x10^{10}$ atoms of ^{14}C per gram. Compute the approximate age of the relic.

$t =$ time in years

$A(t) =$ amt of ^{14}C remaining after t yrs

$A(0) = 5 \times 10^{10}$

$$\frac{dA}{dt} = kA$$

$$\Rightarrow \frac{dA}{A} = k\,dt$$

$$\Rightarrow \ln|A| = kt + c$$

$$\Rightarrow A = A_0 e^{kt}$$

find k

$$\frac{1}{2} = e^{k(5730)}$$

$$\Rightarrow k = \frac{\ln(1/2)}{5730}$$

$$= -1.2097 \times 10^{-4}$$

find age of relic

$$4.6 \times 10^{10} = 5 \times 10^{10} e^{kt}$$

$$\Rightarrow t = \frac{\ln(4.6/5)}{k}$$

$$\approx 690$$

so the relic is about 690 yrs old.

4.) A large tank is filled with 500 gallons of pure water. Brine containing 2 lbs/gal of salt is pumped into the tank at 5 gal/min. The well-mixed solution is pumped out at 10 gal/min. Find the number of lbs of salt in the tank at time *t*.

$t =$ time (min)

$A(t) =$ amt of salt in the tank at time t

$A_0 = 0$

$$\frac{dA}{dt} = \binom{\text{rate}}{\text{in}} - \binom{\text{rate}}{\text{out}}$$

$$= 10\,\frac{\text{lbs}}{\text{min}} - \frac{A}{500-5t}\cdot 10$$

$$\Rightarrow A' = 10 - \frac{2A}{100-t}$$

$$\Rightarrow A' + \frac{2}{100-t}A = 10$$

$$\Rightarrow \frac{1}{(100-t)^2}A' + \frac{2}{(100-t)^3}A = \frac{10}{(100-t)^2}$$

$$\Rightarrow \frac{d}{dt}\left(A\cdot\frac{1}{(100-t)^2}\right) = \frac{10}{(100-t)^2}$$

$$\Rightarrow A\cdot\frac{1}{(100-t)^2} = \frac{10}{100-t} + c$$

$$\Rightarrow A = 10(100-t) + c(100-t)^2$$

$$= 10(100-t) - \frac{1}{10}(100-t)^2$$

Integrating Factor

$$p(t) = \frac{2}{100-t}$$

$$\Rightarrow \int p(t)\,dt = -2\ln|100-t|$$

$$\Rightarrow e^{\int p\,dt} = \frac{1}{(100-t)^2}$$

Solve for c

$$0 = 1000 + c\,10000$$

$$\Rightarrow c = -\frac{1}{10}$$

3.) The rate at which an object's temperature changes is proportional to the difference in temperature between the object and the ambient (surrounding) temperature.

A cake baked in honor of "The Tip" is removed from an oven at 210F and left to cool in room temperature which is 70F. After 30 minutes the cake is 140F. In order to for the cake to have the perfect consistency, it must be eaten at precisely 100F (otherwise it will make its receiver sorry it tasted such a mediocre pastry). When is the ideal time to eat the cake?

$t =$ time out of the oven (min)

$R(t) =$ cake temp at time t.

$R(0) = 210$

ambient temp $T_r = 70$.

$$\frac{dR}{dt} = k(R-70)$$

$$\Rightarrow \frac{dR}{R-70} = k\,dt$$

$$\Rightarrow \ln|R-70| = kt + c$$

$$\Rightarrow R - 70 = ce^{kt}$$

$$\Rightarrow R = 70 + ce^{kt}$$

solve for c.

$R(0) = 210 \Rightarrow c = 140$

solve for k

$$R(30) = 140 = 70 + 140e^{30k}$$

$$\Rightarrow \frac{1}{2} = e^{30k}$$

$$\Rightarrow k = \frac{\ln 1/2}{30}$$

$$= -0.0231$$

find ideal time.

$$100 = 70 + 140e^{kt}$$

$$\Rightarrow \frac{3}{14} = e^{kt}$$

$$\Rightarrow t = \frac{\ln 3/14}{k}$$

$$\approx 67$$

Ideally, the cake should be eaten 67 min after being removed from the oven.