2.5: Solutions by substitutions

There are three types of subs we address in this section.

(1) homogeneous fcn: $f(tx, ty) = t^{\alpha} f(x,y)$ w/ deg. α.

$M(x,y)dx + N(x,y)dy = 0$ is a homogeneous DE if M, N are homogeneous fcns w/ the same degree.

sub: $y = ux$ or $x = vy$.

(2) A Bernoulli DE has the form: $\frac{dy}{dx} + P(x)y = f(x)y^{n}$

sub: $u = y^{1-n}$ for $n \neq 0, 1$ or $y = u^{\frac{1}{1-n}}$

(3) Reduce $\frac{dy}{dx} = f(Ax + By + C)$ to separation of variables.

sub: $u = Ax + By + C$

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ex1: $t^2 \frac{dy}{dt} + y^2 = ty$

$$\Rightarrow \frac{dy}{dt} - \frac{1}{t} y = -\frac{1}{t^2} y^2 \qquad \text{Bernoulli D.E.}$$

Let $u = y^{-1}$ OR $y = u^{-1}$

$$\Rightarrow \frac{dy}{dt} = -u^{-2}\frac{du}{dt}$$

$$\Rightarrow -u^{-2}\frac{du}{dt} - \frac{1}{t}u^{-1} = -\frac{1}{t^2}u^{-2}$$

$$\Rightarrow \frac{du}{dt} + \frac{1}{t}u = \frac{1}{t^2}$$

$$\Rightarrow t u' + u = \frac{1}{t}$$

$$\Rightarrow \frac{d}{dt}(tu) = \frac{1}{t}$$

$$\Rightarrow tu = \ln|t| + c$$

$$\Rightarrow u = \frac{1}{t}\ln|t| + \frac{c}{t}$$

$$\Rightarrow \frac{1}{y} = \frac{1}{t}\ln|t| + \frac{c}{t}$$

$$\Rightarrow y = \frac{t}{\ln(t) + c}$$

$p(t) = \frac{1}{t}$

$\int p(t)\,dt = \ln|t|$

$e^{\int p(t)dt} = |t|$

for $t > 0$, $|t| = t$

ex 2: $y\,dx + x(\ln x - \ln y - 1)\,dy = 0$, $y(1) = e$

Homogeneous w/ degree : $\alpha = 1$.

Let $y = ux$ $\Rightarrow$ $dy = x\,du + u\,dx$

so $ux\,dx + x(\underbrace{\ln x - \ln(ux) - 1}_{-\ln u - 1})(x\,du + u\,dx) = 0$

$\Rightarrow ux\,dx + x(-\ln u - 1)(x\,du + u\,dx) = 0$

$\Rightarrow ux\,dx + x^2(-\ln u - 1)\,du + xu(-\ln u - 1)\,dx = 0$

$\Rightarrow -xu\ln u\,dx + x^2(-\ln u - 1)\,du = 0$

$\Rightarrow +\frac{1}{x}\,dx = -\frac{\ln u + 1}{u\ln u}\,du$

$\Rightarrow \ln|x| = -\ln|u\ln u| + c$

$\Rightarrow \ln|x| = -\ln\left|\frac{y}{x}\ln\frac{y}{x}\right| + c$

To solve for c: $(1, e)$

$0 = -\ln|e\ln(e)| + c$

$= -1 + c$

$\Rightarrow c = 1$

hence $\ln|x| = 1 - \ln\left|\frac{y}{x}\ln\frac{y}{x}\right|$

note: this is clearer if $x = vy$.
$y\ln\left|\frac{x}{y}\right| = -e$

$\boxed{\begin{matrix}2.5\\4/4\end{matrix}}$

ex 3: $\frac{dy}{dx} = \tan^2(x+y)$

$\frac{s^2}{c^2} + \frac{c^2}{c^2} = \frac{1}{c^2}$

Let $u = x + y$

$\frac{du}{dx} = 1 + \frac{dy}{dx}$

$\Rightarrow \frac{du}{dx} - 1 = \tan^2 u$

$\Rightarrow \frac{du}{dx} = \tan^2 u + 1$

$\Rightarrow \frac{du}{dx} = \sec^2 u$

$\Rightarrow \cos^2 u \, du = dx$

$\Rightarrow \frac{1}{2}u + \frac{1}{4}\sin 2u = x + c$

$\Rightarrow 2(x+y) + \sin(2(x+y)) = 4x + c$