2.4: Exact Equations

__ex1:__ a clever way to solve $\frac{dy}{dx} = -\frac{2x+y}{x+6y}$

it isn't separable or linear ... try:

$-(x+6y)dy = (2x+y)dx$

$\Rightarrow \underbrace{(2x+y)}_{M(x,y)}dx + \underbrace{(x+6y)}_{N(x,y)}dy = 0$

Suppose there was some fct $f(x,y)$ s.t. this DE is the differential of f.

Let's try to find f:

$f(x,y) = \int \overbrace{(2x+y)}^{M} dx$

$= x^2 + xy + g(y)$

Now $\frac{\partial f}{\partial y} = x + g'(y) = x + 6y$

$\Rightarrow g'(y) = 6y$ and $g(y) = 3y^2 + c$

So $f(x,y) = x^2 + xy + 3y^2 + c$

and $x^2 + xy + 3y^2 + c = 0$ is a soln to the DE.

__check__ $\frac{d}{dx}(x^2 + xy + 3y^2 + c) = \frac{d}{dx}(0)$

$\Rightarrow 2x + y + xy' + 6yy' = 0$

$\Rightarrow y' = -\frac{2x+y}{x+6y}$ ✓

In the last example, we referenced differentials.

Recall from calculus:

(A) If $y = f(x)$, the differential is:

$$dy = f'(x)\,dx \quad \text{(section 3.10)}$$

or $$dy = \frac{dy}{dx}\,dx$$

(B) If $z = f(x,y)$, the differential is:

$$dz = f_x(x,y)\,dx + f_y(x,y)\,dy$$

OR $$dz = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy \quad \text{(section 14.4)}$$

(C) If $0 = f(x,y)$

then $$0 = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy$$

Dfn: A differential expression $M(x,y)\,dx + N(x,y)\,dy$ is an <u>exact differential</u> in a region R of the xy-plane if it corresponds to the differential of some fcn $f(x,y)$ defined on R.

Dfn: A 1st-order DE of the form $M(x,y)\,dx + N(x,y)\,dy = 0$ is an <u>exact eqt</u> if the LHS is an exact differential.

ex2: Suppose $$(\underbrace{6xy - y^3}_{M(x,y)})dx + (\underbrace{4y + 3x^2 - 3xy^2}_{N(x,y)})dy = 0$$ is an exact eqt. Find the solution $f(x,y) = 0$ s.t. $f_x = M$ and $f_y = N$.

$$f(x,y) = \int (6xy - y^3)\,dx$$

$$= 3x^2y - y^3x + g(y).$$

Now $$\frac{\partial f}{\partial y} = 3x^2 - 3xy^2 + g'(y) = 4y + 3x^2 - 3xy^2$$

↲

$\hookrightarrow$ $g'(y) = 4y$

$\Rightarrow g(y) = 2y^2 + c$

so $f(x,y) = 3x^2y - xy^3 + 2y^2 + c$

and our soln. family is $3x^2y - xy^3 + 2y^2 + c = C$

General derivation.

The previous example begins by supposing we have an exact eqt. But how do we know this?

Criterion for an exact eqt:

<u>short</u>: $M(x,y)dx + N(x,y)dy$ is exact iff $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

<u>long</u>: Let $M(x,y)$ and $N(x,y)$ be continuous first partials in a rect. region R. Then a necessary and sufficient condition that $M(x,y)dx + N(x,y)dy$ be an exact differential is $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

proof of necessity

Assume M, N have cont. partials on $\mathbb{R}^2$ so we can ignore R.

If $M(x,y)dx + N(x,y)dy$ is exact, then $\exists f$ s.t. $\frac{\partial f}{\partial x} = M$ and $\frac{\partial f}{\partial y} = N$

$$\Rightarrow \frac{\partial M}{\partial y} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \underbrace{\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}}_{\text{by Clairaut's Thm in 14.2}} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial N}{\partial x}$$

To show sufficiency, we assume $M(x,y)\,dx + N(x,y)\,dy = 0$ and $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ and then show the LHS is an exact differential.

<u>NTS</u>: There is an $f(x,y)$ s.t. LHS is its differential

For such an f to exist

$$\frac{\partial f}{\partial x} = M(x,y)$$

$$\Rightarrow f(x,y) = \int M(x,y)\,dx + g(y)$$

$$\Rightarrow \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\int M(x,y)\,dx + g'(y) = N(x,y)$$

$$* \quad \Rightarrow g'(y) = N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\,dx$$

integrate to find $g(y)$ and the soln is $f(x,y) = c$

The touchy point is $(*)$ where we claim that ugly RHS is a fct of y (and not x). To see this, we differentiate WRT x and show $\frac{\partial}{\partial x}$ RHS $= 0$.

$$\frac{\partial}{\partial x}\left(N(x,y) - \frac{\partial}{\partial y}\int M(x,y)\,dx\right) = \frac{\partial N}{\partial x} - \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\int M(x,y)\,dx\right)$$

$$\longrightarrow \quad = \frac{\partial N}{\partial x} - \frac{\partial}{\partial y}\left(\frac{\partial}{\partial x}\int M(x,y)\,dx\right)$$

Clairaut's Thm and assumption

$$= \frac{\partial N}{\partial x} - \frac{\partial}{\partial y} M(x,y)$$

$$= 0$$

Ex 3: Solve $y\,dx + \sec^2 x\,dy = 0$ (Not exact, but is separable)

Ex4: $(e^x \sin y + \tan y)dx + (e^x \cos y + x\sec^2 y)dy = 0$

$$\frac{\partial M}{\partial y} = e^x \cos y + \sec^2 y = \frac{\partial N}{\partial x} \quad \text{... so exact}$$

$$f(x,y) = \int (e^x \sin y + \tan y)\,dx$$

$$= e^x \sin y + x\tan y + g(y)$$

$$\frac{\partial f}{\partial y} = e^x \cos y + x\sec^2 y + g'(y) = (e^x \cos y + x\sec^2 y) \Rightarrow g'(y) = 0$$

So $e^x \sin y + x\tan y = c$.

Sometimes we can combine the idea of an integrating factor w/ this idea

ex5: $y^2 \cos x\,dx + (4 + 5y\sin x)dy = 0$

... not exact.

$$\frac{M_y - N_x}{N} \quad \text{fct of } x \text{ \& } y \quad \text{(stuck)}$$

$$\frac{N_x - M_y}{M} = \frac{3}{y} \quad \text{fct of } y \ldots \quad \mu(y) = e^{\int \frac{3}{y}dy} = y^3$$

is our integrating factor.