2.3: Linear Equations

Recall: 1st order linear DE

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

$$\Rightarrow \quad \frac{dy}{dx} + P(x)y = f(x) \qquad \text{standard form}$$

Plan: solve a couple examples using a method dependent on the product rule... then we will derive the method.

ex1: $xy' + 2y = 3x$

$$\Rightarrow \frac{dy}{dx} + \frac{2}{x}\cdot y = 3$$

Trick

$P(x) = \frac{2}{x}$

$\int P(x)dx = 2\ln|x|$

$e^{\int P(x)dx} = e^{2\ln|x|}$

$= x^2$

$$\Rightarrow x^2\frac{dy}{dx} + 2xy = 3x^2$$

$$\Rightarrow \frac{d}{dx}(x^2y) = 3x^2$$

$$\Rightarrow x^2y = x^3 + c$$

$$\Rightarrow y = x + \frac{c}{x^2}$$

Method of solution

The LHS of $\frac{dy}{dx} + P(x)y = f(x)$

can be massaged into the exact derivative of a product.

Find the magic μ.

consider $\frac{d}{dx}(\mu(x)y) = \frac{d\mu}{dx}y + \frac{dy}{dx}\mu(x)$

to get to this multiple LHS by $\mu(x)$

$$\mu(x)\left[\frac{dy}{dx} + P(x)y\right] = \mu(x)\frac{dy}{dx} + \mu(x)P(x)y$$

$$= \mu(x)\frac{dy}{dx} + \frac{d\mu}{dx}y$$

this works if $\mu(x)P(x) = \frac{d\mu}{dx}$. This is separable.

$\Rightarrow \quad P(x)\,dx = \frac{d\mu}{\mu}$

$\Rightarrow \quad \int P(x)dx + c = \ln|\mu|$

$\Rightarrow \quad |\mu| = ke^{\int P(x)dx}$

$\Rightarrow \quad \mu(x) = ke^{\int P(x)dx}$ (note: $k = e^c > 0$)

So, by choosing the right integrating factor w/which to multiply both sides of the DE, we can write the sum as the derivative of a product.

2.3
3/4

ex 2: $y' + 2xy = x \ , \ y(0) = -2$

Trick

$p(x) = 2x$

$\int p(x)\,dx = x^2$

$e^{\int p(x)\,dx} = e^{x^2}$

$\Rightarrow e^{x^2} y' + 2x e^{x^2} y = x e^{x^2}$

$\Rightarrow \frac{d}{dx}\left(y e^{x^2}\right) = x e^{x^2}$

$\Rightarrow y e^{x^2} = \frac{1}{2} e^{x^2} + c$

$\Rightarrow y = \frac{1}{2} + c e^{-x^2}$ and since $y(0) = -2$

$\Rightarrow y = \frac{1}{2} - \frac{5}{2} e^{-x^2}$

$x\frac{dy}{dx} = 4y \Rightarrow \frac{dy}{dx} - \frac{4}{x}y = 0$ is an example of a homogeneous 1st order linear DE. It can be solved by separating variables and by multiplying by the integration factor.

$x\frac{dy}{dx} = 4y + 3x \Rightarrow \frac{dy}{dx} - \frac{4}{x}y = 3x$ is a non homogeneous 1st order linear DE that can be solved w/ the integrating method but not separation of variables.

ex3: $y\,dx = (y^2 - xy^2)\,dy$, $x(0) = 3$ $\quad P(y) = y$

$\Rightarrow \frac{dx}{dy} = y - xy \qquad \int y\,dy = \frac{y^2}{2}$

$\Rightarrow \frac{dx}{dy} + xy = y \qquad e^{\int y\,dy} = e^{\frac{1}{2}y^2}$

$\Rightarrow e^{\frac{1}{2}y^2}x' + y e^{\frac{1}{2}y^2}x = y e^{\frac{1}{2}y^2}$

$\Rightarrow \frac{d}{dy}\left(e^{\frac{1}{2}y^2}x\right) = ye^{\frac{1}{2}y^2}$

$\Rightarrow e^{\frac{1}{2}y^2}x = e^{\frac{1}{2}y^2} + c$

$\Rightarrow x = 1 + ce^{-\frac{1}{2}y^2}$ and $c = 2$

This can also be solved using using separation of variables. The abs can be resolved because of the discontinuity of ln.

so $x(y) = 1 + 2e^{-\frac{1}{2}y^2}$

we call $2e^{-\frac{1}{2}y^2}$ a "transient term."