1.3: DE as mathematical models

The books description of mathematical modeling (p20-1) was excellent ... especially as it discussed what it called the level of resolution of the model.

For example, a low-res pop model is based on the premise that the rate @ which a pop. grows is <u>proportional</u> pop. at that time.

t = time. $\quad \propto$ means proportional

$P(t)$ = pop @ time t

$\frac{dP}{dt}$ = RoC of the pop.

so $\frac{dP}{dt} \propto P(t)$ or $\frac{dP}{dt} = k P(t)$

Q: What are a few variables that this model fails to address?

If $A(t)$ gives the # of nuclei of a radioactive substance remaining @ time t

$$\frac{dA}{dt} \propto A(t) \quad \text{OR} \quad \frac{dA}{dt} = kA(t)$$

If $S(t)$ gives the amount of $ in an acct that earns an annual interest rate of r compounded cont.

$$\frac{dS}{dt} = rS$$

If $D(t)$ gives the amt of a med in the bloodstream t hrs after taking the med.

$$\frac{dD}{dt} = kD$$

Note: A single DE can serve as a math. model for many different phenomena.

This is an example of the unifying nature of mathematics.

It is one of the great mysteries ... read more in Wigner's 1960 article.

ex 1: Set up a model for a pop $P(t)$ is the birth rate is prop. to the pop. but the death rate is prop. to the square of the pop.

ex 2: Set up a model to describe the temp of a mass moved from the refrig. to the oven.

$$\frac{dT}{dt} \propto T - T_m$$

T_m = oven temp

$T(0)$ = fridge temp

ex3: A large tank is filled w/ 500 gal. of pure water. Brine containing 2 lbs/gal of salt is pumped in @ 5 gal/min. The well mixed soln. is pumped out @ the same rate. Set up a DE to describe the situation.

t = time in min

A(t) = amt of salt (in lbs) in the tank at time t.

$$\frac{dA}{dt} = \begin{pmatrix}\text{rate salt} \\ \text{enters}\end{pmatrix} - \begin{pmatrix}\text{rate salt} \\ \text{exits}\end{pmatrix}$$

$$= 10 - \frac{A(t)}{500} \cdot 5$$

ex4: Same as (ex3), but brine is pumped out at 7 gal/min.

$$\frac{dA}{dt} = \begin{pmatrix}\text{rate salt} \\ \text{enters}\end{pmatrix} - \begin{pmatrix}\text{rate salt} \\ \text{exits}\end{pmatrix}$$

$$= 10 - \frac{A(t)}{500 - 2t} \cdot 7$$

1.3: DE's as Mathematical Models (a few examples)

1.) Set up a model for a population $P(t)$ if the birth rate is proportional to the population at time t but the death rate is proportional to the square of the population.

2.) Set up a model to describe the temperature of a mass moved from the refrigerator to the oven

3.) A large tank is filled with 500 gallons of pure water. Brine containing 2 lbs/gal of salt is pumped into the tank at a rate of 5 gal/min. The well-mixed solution is pumped out at the same rate. Set up a differential equation to model the amount of salt in the tank at time t.

4.) The situation is the same as above, but the mixture is pumped out at a rate of 7 gal/min.

Mathematical Models It is often desirable to describe the behavior of some real-life system or phenomenon, whether physical, sociological, or even economic, in mathematical terms. The mathematical description of a system of phenomenon is called a **mathematical model** and is constructed with certain goals in mind. For example, we may wish to understand the mechanisms of a certain ecosystem by studying the growth of animal populations in that system, or we may wish to date fossils by analyzing the decay of a radioactive substance, either in the fossil or in the stratum in which it was discovered.

Construction of a mathematical model of a system starts with

(*i*) identification of the variables that are responsible for changing the system. We may choose not to incorporate all these variables into the model at first. In this step we are specifying the **level of resolution** of the model.

Next

(*ii*) we make a set of reasonable assumptions, or hypotheses, about the system we are trying to describe. These assumptions will also include any empirical laws that may be applicable to the system.

For some purposes it may be perfectly within reason to be content with low-resolution models. For example, you may already be aware that in beginning physics courses, the retarding force of air friction is sometimes ignored in modeling the motion of a body falling near the surface of the Earth, but if you are a scientist whose job it is to accurately predict the flight path of a long-range projectile, you have to take into account air resistance and other factors such as the curvature of the Earth.

Since the assumptions made about a system frequently involve *a rate of change* of one or more of the variables, the mathematical depiction of all these assumptions may be one or more equations involving *derivatives*. In other words, the mathematical model may be a differential equation or a system of differential equations.

Once we have formulated a mathematical model that is either a differential equation or a system of differential equations, we are faced with the not insignificant problem of trying to solve it. *If* we can solve it, then we deem the model to be reasonable if its solution is consistent with either experimental data or known facts about the behavior of the system. But if the predictions produced by the solution are poor, we can either increase the level of resolution of the model or make alternative assumptions about the mechanisms for change in the system. The steps of the modeling process are then repeated, as shown in the diagram in Figure 1.3.1.

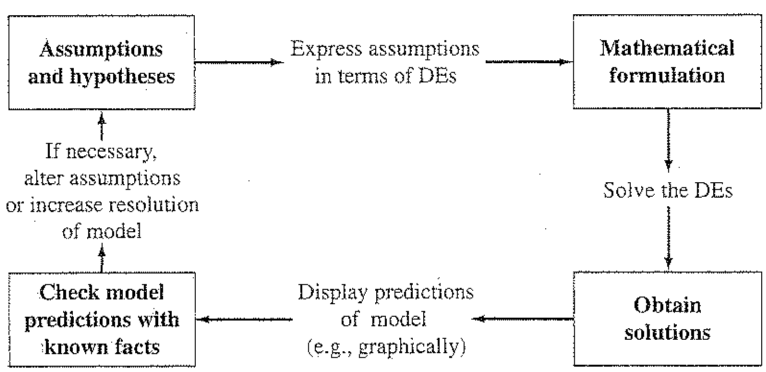

FIGURE 1.3.1 Steps in the modeling process with differential equations

Of course, by increasing the resolution, we add to the complexity of the mathematical model and increase the likelihood that we cannot obtain an explicit solution.

A mathematical model of a physical system will often involve the variable time t. A solution of the model then gives the **state of the system**; in other words, the values of the dependent variable (or variables) for appropriate values of t describe the system in the past, present, and future.

Population Dynamics One of the earliest attempts to model human **population growth** by means of mathematics was by the English clergyman and economist Thomas Malthus in 1798. Basically, the idea behind the Malthusian model is the assumption that the rate at which the population of a country grows at a certain time is