Definition: A rational expression is an expression consisting of a polynomial expression by a polynomial expression.

Example 1:

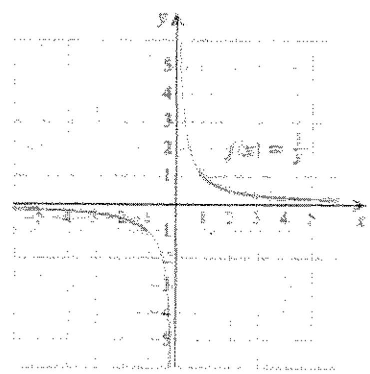
$$\frac{3x^2-2}{x+1} \qquad \frac{5x^7+2x+7}{4-x+x^3}$$

$$\frac{1}{2x+1} \qquad 2x+1 = \frac{2x+1}{1}$$

Rational expressions can be used to form functions called rational functions.

Example 2: $f(x)=\frac{1}{x}$ ← rational function.

↑ function



As x gets big ($+\infty$), $y \to 0$

As $x \to -\infty$, $y \to 0$

As $x \to 0$ (positive x-values), $y \to +\infty$

As $x \to 0$ (negative x-vals), $y \to -\infty$

summary:

$\frac{1}{\text{big}} \approx 0$ but $\frac{1}{0} \approx \pm\infty$ (Approaches)

can't divide by zero

* $\frac{1}{0}$ is undefined

Example 3: Rik usually takes 3 hours more than Pearl does to process a day's orders at Liberty Place Photo. If Pearl takes *t* hours to process a day's orders, the function given by $H(t) = \frac{t^2 + 3t}{2t + 3}$ can be used to determine how long it would take if they worked together.

How long will it take them, working together to complete a day's orders if Pearl can process the orders alone in 5 hours?

t = time for Pearl working alone.

H(t) = time if they worked together.

So $t = 5$: $H(5) = \frac{25 + 15}{10 + 3} = \frac{40}{13}$ hrs.

working together, they can complete the job in $\frac{40}{13}$ hrs.

Method: Products of Rational Expressions

To multiply two rational expressions, multiply numerators and multiply denominators:

$$\frac{A}{B} \cdot \frac{C}{D} = \frac{AC}{BD}, \text{ where } B \neq 0, D \neq 0$$

Example 4: Multiply $\frac{x+5}{x-4} \cdot \frac{y-1}{x^3}$ $= \frac{(x+5)(y-1)}{x^3(x-4)}$

Review: How do we reduce $\frac{8}{14}$?

$$\frac{8}{14} = \frac{\cancel{2}\cdot 4}{\cancel{2}\cdot 7} = \frac{4}{7}$$

Similarly, reduce $\frac{(x-3)\cancel{(x-2)}}{(x+5)\cancel{(x-2)}}$ $= \frac{x-3}{x+5}$

What is different between $\frac{(x-3)(x-2)}{(x+5)(x-2)}$ and $\frac{(x-3)}{(x+5)}$?

$x \neq -5$ and $x \neq 2$ $\qquad$ $x \neq -5$

when we cancel we can lose information.

Example 5: Write the function in simplified form. Be careful with the domains

a.) $f(t) = \frac{5t^2+20t}{t^2+4t} = \frac{5\cancel{t}\cancel{(t+4)}}{\cancel{t}\cancel{(t+4)}} = \frac{5}{1} = 5$, $t \neq 0$ and $t \neq -4$

b.) $g(m) = \frac{m^2-9}{3m+3} \cdot \frac{m+3}{m-3} = \frac{(m+3)\cancel{(m-3)}(m+3)}{3(m+1)\cancel{(m-3)}}$

$$= \frac{(m+3)^2}{3(m+1)}, \ m \neq 3$$

Example 6: Simplify

a.) $\frac{x^2-16}{x^2}\cdot\frac{x^2-4x}{x^2-x-12}$ $=$ $\frac{(x+4)(x-4)\cdot\cancel{x}\cancel{(x-4)}}{\cancel{x^2}\,\cancel{(x-4)}(x+3)}$

$= \frac{(x+4)(x-4)}{x(x+3)}$, $x \neq 4$

b.) $\frac{a^2-1}{2-5a}\cdot\frac{15a-6}{a^2+5a-6}$ $=$ $\frac{(a+1)\cancel{(a-1)}\cdot 3\cancel{(5a-2)}}{-\cancel{(5a-2)}\,(a+6)\cancel{(a-1)}}$

$= -\frac{3(a+1)}{a+6}$, $a \neq 1$ and $a \neq \frac{2}{5}$

$5a-2=0$

$5a=2$

$a=2/5$

Important: We canNOT cancel over Addition/ Subtraction

Example 7: Simplify

a.) $\frac{x+1}{x}$ $\neq 1$. $\frac{x+1}{x} = \frac{x}{x} + \frac{1}{x} = 1 + \frac{1}{x}$

b.) $\frac{6t-1}{2} = \frac{6t}{2} - \frac{1}{2} = 3t - \frac{1}{2}$

c.) $\frac{2x}{x+1}$ yeah! already done.

d.) $\dfrac{x^3+y^3}{x^2+2xy-3y^2}\cdot\dfrac{x^2-y^2}{3x^2+6xy+3y^2}$ $\leftarrow$ $3(x^2+2xy+y^2)$

$$= \frac{\cancel{(x+y)}(x^2-xy+y^2)\cancel{(x+y)}\cancel{(x-y)}}{\cancel{(x-y)}(x+3y)\cdot 3\cancel{(x+y)^2}}$$

$$= \frac{x^2-xy+y^2}{3(x+3y)}$$

Method: Quotients of Rational Expressions

To divide two rational expressions, invert the second expression and multiply:

$$\frac{A}{B}\div\frac{C}{D}=\frac{A}{B}\cdot\frac{D}{C}=\frac{AD}{BC}, \text{where} B\neq 0, D\neq 0$$

Example 8: Simplify

a.) $\dfrac{3y+15}{y^7}\div\dfrac{y+5}{y^2}$ $= \dfrac{3y+15}{y^7}\times\dfrac{y^2}{y+5}$

$$= \frac{3\cancel{(y+5)}\cancel{y^2}}{\cancel{y^7}^{5}\cancel{(y+5)}}$$

$$= \frac{3}{y^5}, \ y\neq -5$$

b.) $\dfrac{x^2-y^2}{4x+4y}\div\dfrac{3y-3x}{12x^2}$

$$= \frac{(x^2-y^2)\cdot 12x^2}{(4x+4y)\cdot(3y-3x)}$$

$$= \frac{(x-y)\cancel{(x+y)}\cdot\cancel{12}x^2}{\cancel{4}\cancel{(x+y)}\cancel{3}(y-x)}$$

$$= \frac{x^2(x-y)}{y-x} = \frac{x^2\cancel{(x-y)}}{-\cancel{(x-y)}} = \frac{x^2}{-1} = -x^2$$

Example 9: Simplify $g(x)=\dfrac{x^2-9}{x^2}\div\dfrac{x^5+3x^4}{x+2}$ and list all domain restrictions

$$= \frac{(x-3)(x+3)}{x^2}\div\frac{x^4(x+3)}{x+2}$$

$$= \frac{(x-3)\cancel{(x+3)}(x+2)}{x^2\cdot x^4\cancel{(x+3)}}$$

$$= \frac{(x-3)(x+2)}{x^6}, \ x\neq -3, 0, -2$$

Let's explore an example to learn a bit about vertical asymptotes ... consider $H(t)=\frac{t^2+5t}{2t+5}$. Use your calculator to generate a graph. Looking at the graph, what happens at $t=-\frac{5}{2}$? This is called a vertical asymptote.

vertical line the graph approaches but does not touch

Chris: "imaginary not line" called this

Example 10: Consider $f(x)=\frac{(x-1)(x+3)}{(2x+1)(x+3)}$ and $g(x)=\frac{x-1}{2x+1}$. Find and compare their vertical asymptotes and domains.

vertical asymptote at $x=-\frac{1}{2}$

hole @ $x=-3$

Domain of f

$\mathbb{R}$ and not $x=-\frac{1}{2},-3$

$(-\infty,-3)\cup(-3,-\frac{1}{2})\cup(-\frac{1}{2},\infty)$

Domain of g

$\mathbb{R}$ and not $x=-\frac{1}{2}$

$(-\infty,-\frac{1}{2})\cup(-\frac{1}{2},\infty)$

Example 11: Find the vertical asymptote(s) of $g(x)=\frac{x^2-4}{2x^2-5x+2}$.

vertical asymptote @ $x=\frac{1}{2}$

hole @ $x=2$

$$=\frac{(x+2)\cancel{(x-2)}}{(2x-1)\cancel{(x-2)}}$$

$$=\frac{x+2}{2x-1},\quad x\neq 2$$