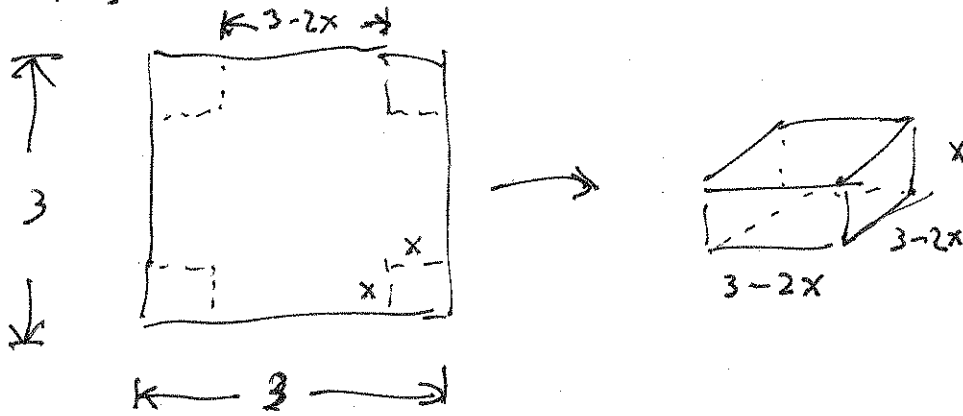**Optimization**

<u>Example 1</u>: A box with an open top is to be constructed from a square piece of cardboard, 3 feet wide, by cutting out a square from each of the four corners and bending up the sides. Find the largest volume that such a box can have.
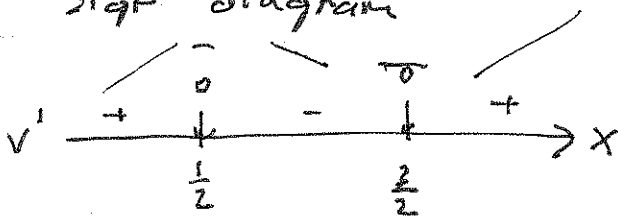
(1) Draw a pic.

(2) Set up an eqt.

$$V(x) = x(3-2x)^2$$
$$= x(9 - 12x + 4x^2)$$
$$= 4x^3 - 12x^2 + 9x$$

(3) Take the deriv.

$$V'(x) = 12x^2 - 24x + 9$$
$$= 3(4x^2 - 8x + 3)$$
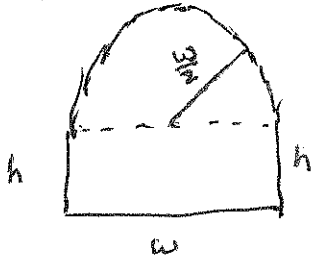$$= 3(2x - 1)(2x - 3)$$

(4) Sign diagram

max at $(\frac{1}{2}, 2)$

(5) The max volume is $2 \, ft^3$.

Example 2: A Norman window has the shape of a rectangle surmounted by a semicircle. If the perimeter of the window is 30 feet, find the dimensions of the window so that the greatest possible amount of light is admitted.

(1) Draw a pic

(2) Set up an eqt.

perimeter: $2h + w + \pi \cdot \frac{w}{2} = 30 \Rightarrow h = \dfrac{30 - \frac{\pi \cdot w}{2} - w}{2}$

area: $A = hw + \dfrac{\pi \cdot \left(\frac{w}{2}\right)^2}{2}$

$= \dfrac{30 - \frac{\pi \cdot w}{2} - w}{2} \cdot w + \dfrac{\pi w^2}{8}$

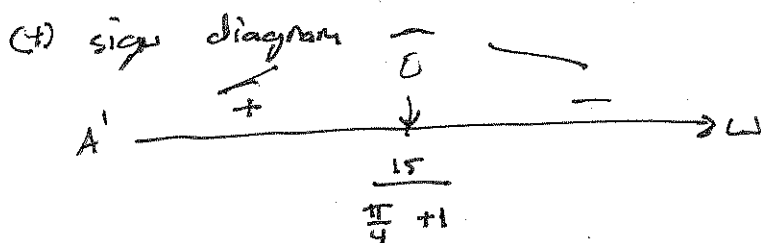$= 15w - \dfrac{\pi w^2}{4} - \dfrac{w^2}{2} + \dfrac{\pi w^2}{8}$

$= 15w + w^2 \left(\dfrac{-\pi}{4} - \dfrac{1}{2} + \dfrac{\pi}{8} \right)$

$= 15w + \left(-\dfrac{\pi}{8} - \dfrac{1}{2} \right) w^2$

(3) Take deriv.

$A' = 15 - 2 \left(\dfrac{\pi}{8} + \dfrac{1}{2} \right) w$

zero @ $w = \dfrac{15}{2 \left(\frac{\pi}{8} + \frac{1}{2} \right)}$

(4) sign diagram

$A' \xrightarrow{\quad + \qquad \downarrow \qquad - \qquad} w$

$\dfrac{15}{\frac{\pi}{4} + 1}$

Max @ $w = \dfrac{15}{\frac{\pi}{4} + 1} \approx 8.40$

(5) Answer.

The optimal window has width 8.4 ft and height of 4.2 ft.

Solution 1: using right triangle trig.

Example 3: A woman is at the boat launch of a circular lake with radius 2 miles that has a sidewalk all the way around it. She wants to arrive at the point diametrically opposite the launch (on the other side of the lake) in the shortest possible time. She can walk at the rate of 4 mph and row a boat at 2mph. How should she proceed?

(1) draw a pic
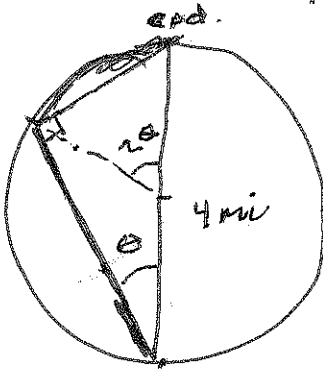
boat only ↓

walk only ↖

end.

2θ

4 mi

θ

start

note: $0 \le \theta \le \dfrac{\pi}{2}$

distance by boat $= 4\cos\theta$

distance by foot $= 2 \cdot 2\theta = 4\theta$

(2) set up eqt.
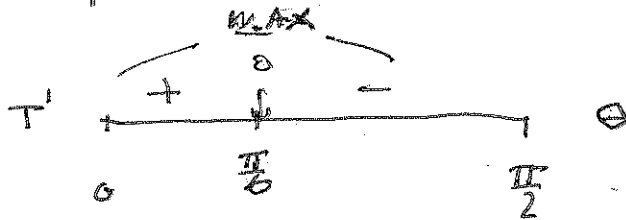
$$T(\theta) = \frac{4\cos\theta}{2} + \frac{4\theta}{4}$$

$$= 2\cos(\theta) + \theta$$

(3) differentiate

$$T'(\theta) = 1 - 2\sin\theta$$

(4) optimize

MAX

θ

T' + + − θ

0   $\dfrac{\pi}{6}$ $\dfrac{\pi}{2}$
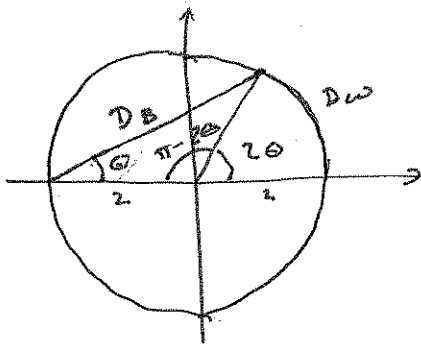
check endpoints: $T(0) = 2$

$T\left(\dfrac{\pi}{2}\right) = \pi/2$ min.

(5) Ans. the question

she should walk.

Solution 2: Using the law of cosines

Example 3: A woman is at the boat launch of a circular lake with radius 2 miles that has a sidewalk all the way around it. She wants to arrive at the point diametrically opposite the launch (on the other side of the lake) in the shortest possible time. She can walk at the rate of 4 mph and row a boat at 2mph. How should she proceed?

$$D_w = 2 \cdot 2\theta$$

$$D_B = \sqrt{8 - 8\cos(\pi - 2\theta)}$$

$$= 2\sqrt{2 - 2\cos(\pi - 2\theta)}$$

and $\quad 0 \le \theta \le \dfrac{\pi}{2}$

So $\quad T(\theta) = \dfrac{D_w}{4} + \dfrac{D_B}{2}$

$$= \theta + \sqrt{2 - 2\cos(\pi - 2\theta)}$$

$\Rightarrow \quad T'(\theta) = 1 + \dfrac{+2\sin(\pi - 2\theta)(-2)}{2\sqrt{2 - 2\cos(\pi - 2\theta)}}$

$$= 1 - \dfrac{2\sin(\pi - 2\theta)}{\sqrt{2 - 2\cos(\pi - 2\theta)}}$$

and solve $\quad T'(\theta) = 0$

$$\dfrac{1}{2} = \dfrac{\sin(\pi - 2\theta)}{\sqrt{2 - 2\cos(\pi - 2\theta)}}$$

$\Rightarrow \quad \dfrac{1}{4} = \dfrac{\sin^2(\pi - 2\theta)}{2 - 2\cos(\pi - 2\theta)}$

$\Rightarrow \quad \dfrac{1}{4}(2 - 2\cos(\pi - 2\theta)) = 1 - \cos^2(\pi - 2\theta)$

$\Rightarrow \quad \cos^2(\pi - 2\theta) - \dfrac{1}{2}\cos(\pi - 2\theta) - \dfrac{1}{2} = 0$

$\Rightarrow \quad 2c^2 - c - 1 = 0 \quad$ where $c = \cos(\pi - 2\theta)$

$\Rightarrow \quad (2c + 1)(c - 1) = 0$

$\Rightarrow \quad c = -\dfrac{1}{2} \quad$ or $\quad c = 1$

So $\cos(\pi - 2\theta) = 1$

$\Rightarrow \pi - 2\theta = 0 \Rightarrow \theta = \dfrac{\pi}{2}$

or

$\pi - 2\theta = 2\pi \Rightarrow \theta = -\dfrac{\pi}{2}$ (outside domain)

or $\cos(\pi - 2\theta) = -\dfrac{1}{2}$

$\Rightarrow \pi - 2\theta = \dfrac{2\pi}{3} \Rightarrow \theta = \dfrac{\pi}{6}$

or

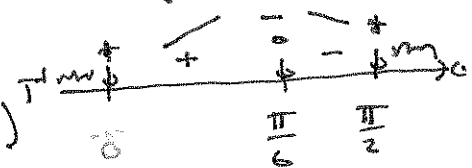$\pi - 2\theta = \dfrac{4\pi}{3} \Rightarrow \theta = -\dfrac{\pi}{6}$ (outside domain)

▽ But T' is undefined when $\cos(\pi - 2\theta) = 1$ or $\theta = \pm\dfrac{\pi}{2}$

▽ Did we pick up other extraneous solutions?

$T'(\dfrac{\pi}{6}) = 0 \quad$ & $\quad T'(-\dfrac{\pi}{6}) = 2$ (extra soln)

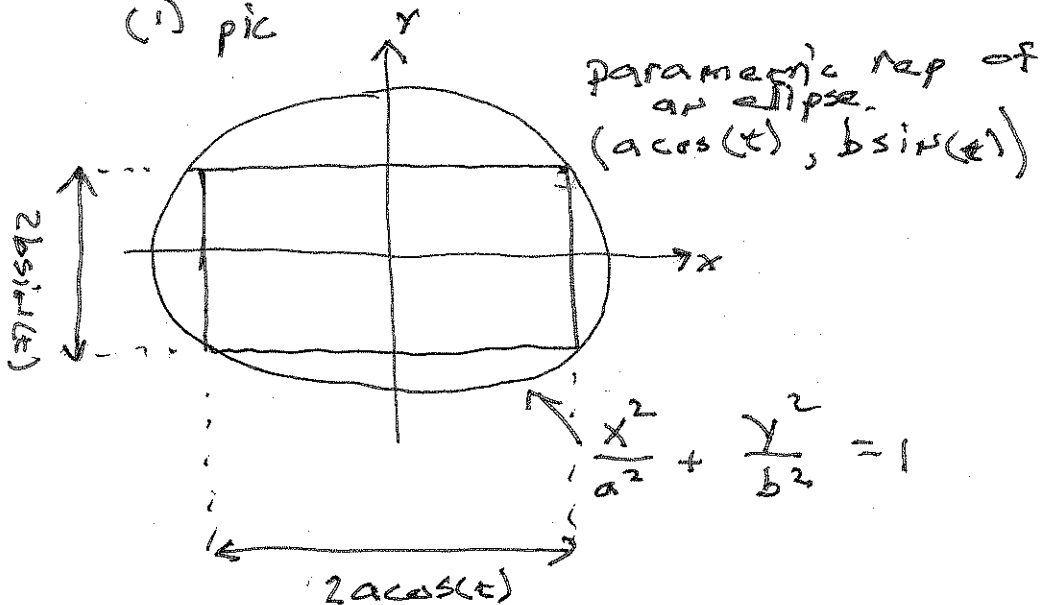There is a max time when $\theta = \dfrac{\pi}{6}$

hmmm.

check endpoints.

$T(0) = 2$ hrs; $T(\dfrac{\pi}{2}) \approx 1.57$

She should walk the whole way.

Solution 1: using parametric equations

Example 4: Find the area of the largest rectangle that can be inscribed in the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$
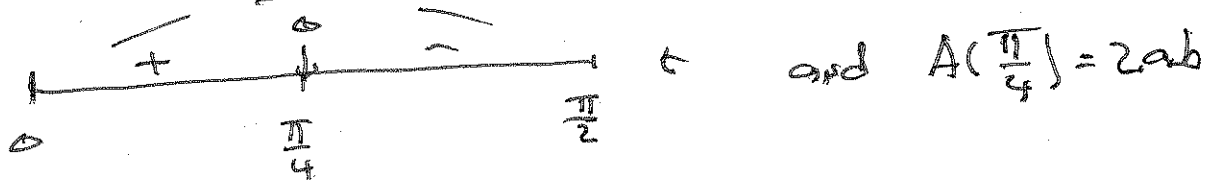
(1) pic

parametric rep of
an ellipse.
$(a\cos(t), b\sin(t))$

$\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

$2b\sin(t)$

$2a\cos(t)$

(2) eqt. for area.

$A(t) = 2a\cos(t) \cdot 2b\sin(t)$

$= 2ab\sin(2t)$

(3) differentiate

$A'(t) = 4ab\cos(2t)$

(4) optimize (only consider $0 \le t \le \pi/2$)

Max

$+$ $\frown$ t and $A(\frac{\pi}{4}) = 2ab$
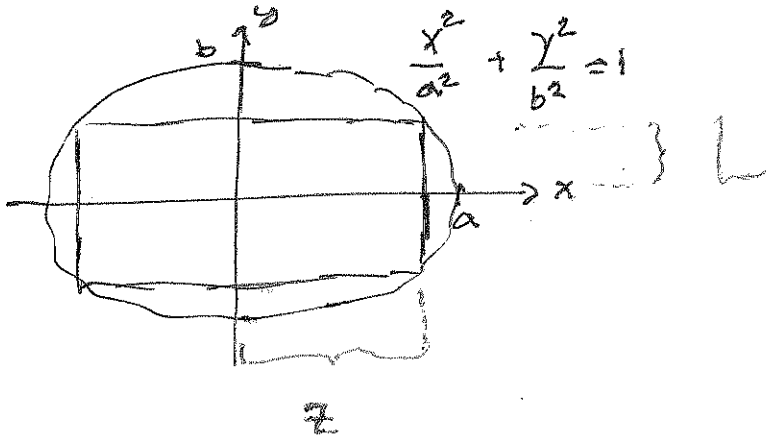
$\frac{\pi}{4}$ $\frac{\pi}{2}$

(5) Ans. the question

The max area is $2ab$.

Solution 2: Using Rectangular Coordinates

Example 4: Find the area of the largest rectangle that can be inscribed in the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

(1) draw a pic.

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \Big\} L$$

(2) set up eqn.

$$Area = (2z)(2L) = 4zL$$

$$\text{and} \quad L = \pm \frac{b}{a}\sqrt{a^2-z^2}$$

$$\text{so} \quad A = 4z \cdot \frac{b}{a}\sqrt{a^2-z^2}$$

(3) differentiate.

$$A' = \frac{4b}{a}\left(1 \cdot \sqrt{a^2-z^2} + z \cdot \frac{-2z}{2\sqrt{a^2-z^2}}\right)$$

$$= \frac{4b}{a} \cdot \frac{(a^2-z^2) - z^2}{\sqrt{a^2-z^2}}$$

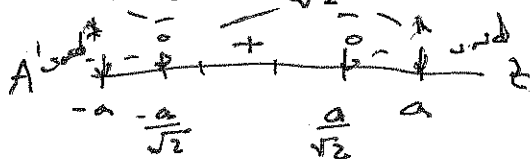$$= \frac{4b}{a} \cdot \frac{a^2 - 2z^2}{\sqrt{a^2-z^2}}$$

(4) find zeros $(A' = 0)$.

solve $a^2 - 2z^2 = 0$

$$\Rightarrow \frac{a^2}{2} = z^2$$

$$\Rightarrow z = \pm \frac{a}{\sqrt{2}}$$

$$A' \text{ sign} \quad \overset{+}{\underset{-a}{}} \overset{-}{\underset{-\frac{a}{\sqrt{2}}}{}} + \overset{-}{} \overset{\text{max}}{\underset{\frac{a}{\sqrt{2}}}{}} \overset{\text{und}}{\underset{a}{}} z$$

(5) maximize area

$$z = \frac{a}{\sqrt{2}}$$

$$A = 4 \cdot \frac{a}{\sqrt{2}} \cdot \frac{b}{a}\sqrt{a^2 - \left(\frac{a}{\sqrt{2}}\right)^2}$$

$$= \frac{4b}{\sqrt{2}}\sqrt{\frac{a^2}{2}}$$
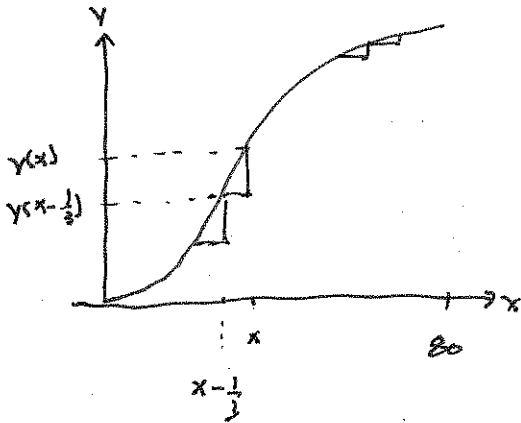
$$= \frac{4ab}{2}$$

$$= 2ab.$$

Solution 1: using a difference

Example 5: A contractor is engaged to build steps up the slope of a hill that has the shape of the graph

$y = \dfrac{x^2(120-x)}{6400}, 0 \le x \le 80$ with x in meters. What is the maximum vertical rise of a stair if each stair

has a horizontal length of one-third meter?

(1) pic

stair he for $h(x) = y(x) - y(x-\frac{1}{3})$

this is valid for $x = \frac{1}{3}, \frac{2}{3}, \ldots, 80$

(2) eqt:

$h(x) = y(x) - y(x-\frac{1}{3})$

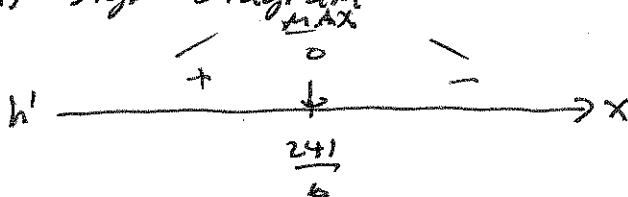$= \dfrac{1}{6400}\left(-x^2 + \dfrac{2169}{27}x - \dfrac{261}{27}\right)$

(I did this in Mathematica)

(3) differentiate

$h'(x) = \dfrac{1}{6400}\left(-2x + \dfrac{2169}{27}\right)$

$h'(x) = 0$ when $x = \dfrac{241}{6}$

(4) sign diagram

The max is at $\frac{241}{6}$ which isn't in the domain. So we check the nearest values 40 & $40\frac{1}{3}$.

$\rightarrow h(40) = h(40\frac{1}{3}) = \dfrac{43199}{172800}$

≈ 0.249994

(5) conclusion.

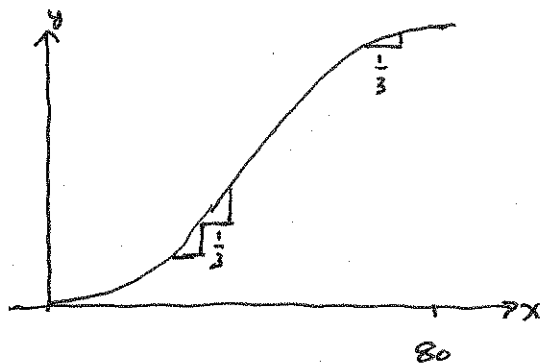The max stair height is about $\frac{1}{4}$ m.

Solution 2: using linearization

Example 5: A contractor is engaged to build steps up the slope of a hill that has the shape of the graph $y = \dfrac{x^2(120-x)}{6400}, 0 \le x \le 80$ with x in meters. What is the maximum vertical rise of a stair if each stair has a horizontal length of one-third meter?

(1) pic

observe: the stair ht
          is most where
          the hill is steepest.

y : hill ht

y' : hill steepness

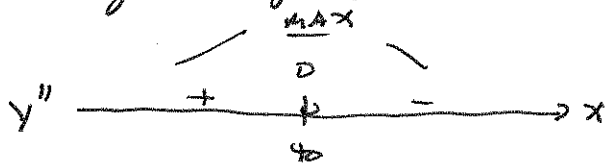y'' : find max hill steepness.

(2) eqt.

$$y = \frac{1}{6400}\left(120x^2 - x^3\right)$$

$$y' = \frac{1}{6400}\left(240x - 3x^2\right)$$

(3) differentiate

$$y'' = \frac{1}{6400}\left(240 - 6x\right)$$

(4) sign diagram

$$\text{max slope} = y'(40) = \frac{3}{4}$$

The run is $\frac{1}{3}$ and so the rise $= \frac{1}{4}$

(5) conclusion!

The max stair height is $\frac{1}{4}$ m.

Example 6: An 8-billion-bushel corn crop brings a price of $2.40/bu. A commodity broker uses the rule of thumb: If the crop is reduced by x percent, then the price increases by 10x cents. Which crop size results in maximum revenue and what is the price per bushel?

(1) Table

q	price	revenue.
8	2.40	$8(2.4)$
−1% 7.92	2.50	$(7.92)(2.5)$
−2% 7.84	2.60	$\vdots$
−3% 7.76	2.70	
$\vdots$		
−x% $8-.08x$	$2.40 + .10x$	$R(x) = (8 - .08x)(2.4 + .10x)$

(2) set up eqr.

$$R(x) = (8 - 0.08x)(2.4 + 0.10x)$$

(3) deriv.
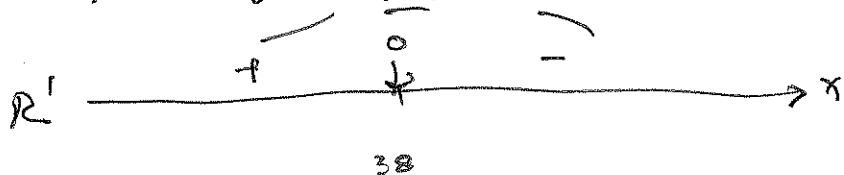
$$R'(x) = -0.08(2.4 + .1x) + .1(8 - 0.08x)$$

$$= -.192 - .008x + .8 - .008x$$

$$= 0.608 - 0.016x$$

$R' = 0$ when $x = 38$
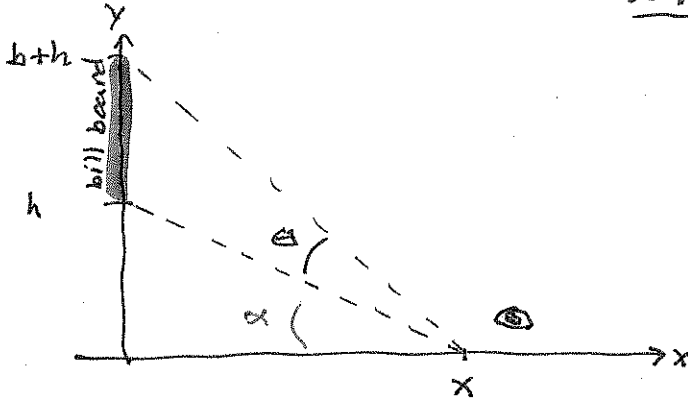
(4) sign diagram

max

R' $\xrightarrow{\qquad + \qquad 0 \qquad - \qquad}$ x

38

(5) our max rev. is when

$8 - 38(.08) = 4.96$ bil. bushels of corn are produced. Each bushel is sold for $6.20/bu.

Example 7: A billboard of height b is mounted on the side of a building with its bottom edge at a distance h from the street below. At what distance x should an observer stand from the wall to maximize the angle of observation?

(1) Draw picture

Solution 1: using arctan and detailed algebra

(2) set up eqt.

$$\tan(\alpha) = \frac{h}{x} \quad \text{and} \quad \tan(\alpha + \theta) = \frac{h+b}{x}$$

$$\Rightarrow \alpha = \arctan\left(\frac{h}{x}\right) \quad \text{and} \quad \alpha + \theta = \arctan\left(\frac{h+b}{x}\right)$$

$$\Rightarrow \theta = \arctan\left(\frac{h+b}{x}\right) - \arctan\left(\frac{h}{x}\right)$$

(3) differentiate

$$\frac{d\theta}{dx} = \frac{1}{1 + \left(\frac{h+b}{x}\right)^2} \cdot \frac{-(h+b)}{x^2} - \frac{1}{1 + \left(\frac{h}{x}\right)^2} \cdot \frac{-h}{x^2}$$

$$= -\frac{x^2}{x^2 + (h+b)^2} \cdot \frac{(h+b)}{x^2} + \frac{x^2}{x^2 + h^2} \cdot \frac{h}{x^2}$$

$$= \frac{h}{x^2 + h^2} - \frac{h+b}{x^2 + (h+b)^2}$$

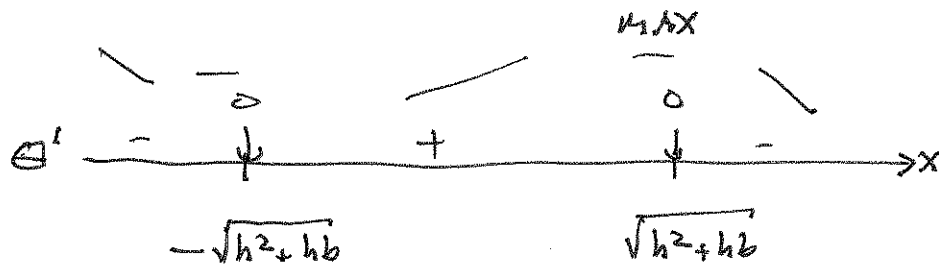$$= \frac{h(x^2 + h^2 + 2hb + b^2) - (h+b)(x^2 + h^2)}{(x^2 + h^2)(x^2 + (h+b)^2)}$$

$$= \frac{b((h^2 + hb) - x^2)}{(x^2 + h^2)(x^2 + (h+b)^2)}$$

$$\begin{aligned}
& hx^2 + h^3 + 2h^2b + hb^2 \\
& -hx^2 - h^3 - bh^2 \qquad - bx^2 \\
\hline
& h^2b + hb^2 - bx^2
\end{aligned}$$

which is $b(h^2 + hb - x^2)$

Note: the denominator > 0 and the numerator has zeros @ $x = \pm\sqrt{h^2 + hb}$

(4) sign diagram

$$\theta' \quad \xrightarrow{\quad - \quad \underset{-\sqrt{h^2+hb}}{\downarrow} \quad + \quad \underset{\sqrt{h^2+hb}}{\downarrow} \quad - \quad } x$$
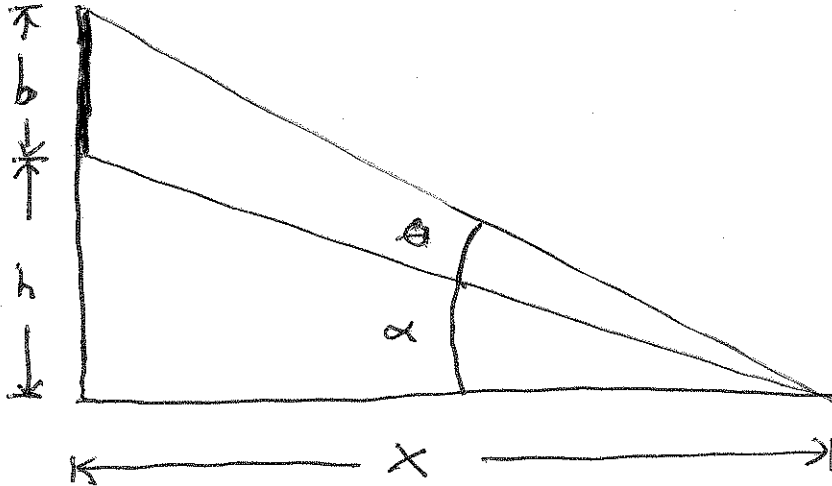
MAX

(5) conclusion

the viewing angle is optimized when the observer is a distance of $\sqrt{h^2+hb}$ from the building.

NOTE: This can also be done using the laws of cosines... but this approach is very painful.

Solution 2: using $\tan(\alpha + \theta)$ identity and easier algebra

(1) pic

(2) eqe.

$$\tan(\alpha + \theta) = \frac{b+h}{x}$$

$$\Rightarrow \frac{b+h}{x} = \frac{\tan\alpha + \tan\theta}{1 - \tan\alpha\,\tan\theta}$$

$$\Rightarrow \frac{b+h}{x} = \frac{\frac{h}{x} + \tan\theta}{1 - \frac{h}{x}\cdot\tan\theta}$$

$$\Rightarrow \frac{b+h}{x}\left(1 - \frac{h}{x}\tan\theta\right) = \frac{h}{x} + \tan\theta$$

$$\Rightarrow \frac{1}{x}\left(b + \cancel{h} - \cancel{h}\right) = \tan\theta\left(1 + \frac{h(b+h)}{x^2}\right)$$

$$\Rightarrow \tan\theta = \frac{\frac{b}{x}}{\frac{x^2 + h^2 + hb}{x^2}}$$

$$\Rightarrow \tan\theta = \frac{xb}{x^2 + h^2 + hb}$$

(3) differentiate implicitly.

$$\sec^2\theta \; \frac{d\theta}{dx} = \frac{b(x^2+h^2+hb) - 2x(xb)}{(x^2+h^2+hb)^2}$$

$$\Rightarrow \quad \frac{d\theta}{dx} = \frac{\cos^2\theta \left(h^2+hb-x^2\right)b}{(x^2+h^2+hb)^2}$$

Notice $0 < \theta < \frac{\pi}{2}$ so $\cos^2\theta \neq 0$
and $(x^2+h^2+hb)^2 \neq 0$ .. both terms
are positive.

(4) optimize

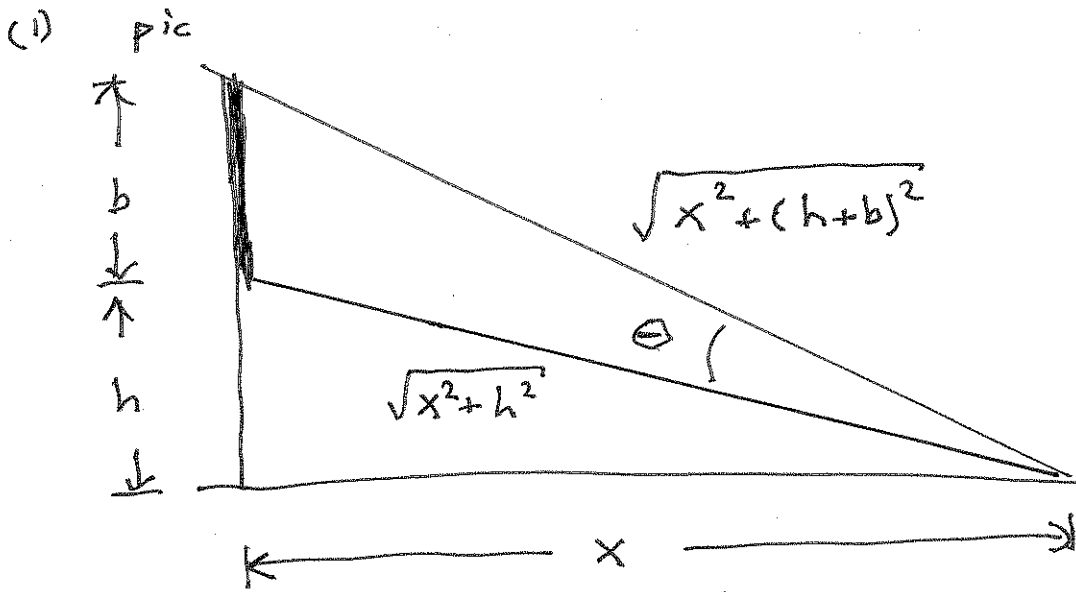$$\frac{d\theta}{dx} = 0 \quad \text{when} \quad h^2 + hb - x^2 = 0$$

$$\Rightarrow \quad x = \pm \sqrt{h^2+hb}$$

(5) Answer the Question.

So the best viewing angle is
when you are $\sqrt{h^2+hb}$ from
the building

solution 3: using the law of cosines.

(1)   pic

$\sqrt{x^2 + (h+b)^2}$

$\sqrt{x^2 + h^2}$

θ

x

(2) eqt (by the law of cosines)

$$b^2 = (x^2 + h^2) + (x^2 + (h+b)^2) - 2\sqrt{x^2+h^2}\sqrt{x^2 + (h+b)^2}\cos\theta$$

$$\Rightarrow \theta = \arccos\left(\frac{2x^2 + 2h^2 + 2hb}{2\sqrt{x^2+h^2}\sqrt{x^2+(h+b)^2}}\right)$$

$$= \arccos\left(\frac{x^2 + h^2 + hb}{\sqrt{x^2+h^2}\sqrt{x^2+(h+b)^2}}\right)$$