1) A Mersenne prime is a prime number of the form $2^p - 1$, where p is prime.

In 1644, Mersenne categorized all numbers of the form $2^n - 1$ as either prime or not, for $n = 1, 2, \ldots, 257$

2) Theorem: If a number of the form $a^n - 1$ is a prime, where $n > 1$ and $a > 0$, then $a = 2$ and n is prime, ie, $a^n - 1 = 2^p - 1$ for some prime p. (So if $a^n - 1$ is prime, it is a Mersenne prime)

[Examples $2^2 - 1 = 3$, $2^3 - 1 = 7$, $2^5 - 1 = 32$, $2^7 - 1 = 127$
$2^{11} - 1 = 2047$ is not, $2^{13} - 1 = 8191 \ldots$]

3) Proof: We assume $a > 0$, $n > 1$, and $a^n - 1$ is prime
Then $a^n - 1 = (a-1)(a^{n-1} + a^{n-2} + \ldots + a + 1)$
Clearly, $a - 1 = 1$, so $a = 2$ (why?)

Also, if n is not prime (ie, is composite) say $n = rs$ with both $r, s > 1$, then

$$2^{rs} - 1 = (2^r - 1)(2^{r(s-1)} + 2^{r(s-2)} + \ldots + 2^r + 1)$$

Since $r, s > 1$, both factors on the right hand side are greater than 1, so $2^{rs} - 1$ would be composite, a contradiction.
So n is prime, $a = 2$, and $a^n - 1 = 2^p - 1$, a Mersenne prime. Q.E.D.

Mersenne Primes have the form $2^p - 1$

M_{48} is the largest known prime.

Claim: If $a^N - 1$ is prime for $a, N \in \mathbb{N}$ and $a > 0$, $N > 1$, then $a = 2$ and N is prime.

Proof.

1st: $a^N - 1 = (a-1)(\underbrace{a^{N-1} + a^{N-2} + \dots + a^2 + a + 1}_{>1})$

$\Rightarrow a - 1 = 1$ since $a^N - 1$ is prime.

$\Rightarrow a = 2$.

2nd: NTS N is prime.

Assume N is not prime. That is

$N = r \cdot s$ where $r, s \in \mathbb{N}$ and $r, s > 1$.

$\Rightarrow a^r - 1 = 2^r - 1$

$= 2^{rs} - 1$

$= (\underbrace{2^r - 1}_{>1})(2^{r(s-1)} + 2^{r(s-2)} + \dots + 2^{2r} + 2^r + 1)$

$\Rightarrow 2^{r(s-1)} + 2^{r(s-2)} + \dots + 2^r + 1 = 1$ (since $a^r - 1$ is prime).

$\Rightarrow \Leftarrow$

Hence N is prime. and $a = 2$.

Q.E.D.

That is, the only primes of the form $a^r - 1$ are Mersenne primes $2^p - 1$