A FEW WORDS FOR THE READER

Many students have difficulty when they are first asked to prove theorems in mathematics. Part of this difficulty may come from an unfamiliarity with the mathematical objects involved (vectors, bases, linear transformations, groups, homomorphisms, and so forth), but a major part of the difficulty seems to be due to an imprecise knowledge of the fundamentals of mathematics: logic, sets, relations and functions. This book attempts to address this problem by giving a concise account of a minimal amount of this material needed to progress further in mathematics and then using this material as a vehicle for gaining practice in proving theorems.

The key word here is *practice*. As you no doubt have observed, learning how to write out a correct proof yourself is quite a bit different from watching someone else write out a proof and understanding that his or her proof is correct. Mathematics is not a spectator sport! Practice and involvement are essential. If anything is to be gained from this book, the reader must become actively engaged in working his or her way through it. This means marking up the pages with questions about unclear passages (should there be any!), doing the examples and then checking the results, working all the exercises and, above all, approaching the subject matter with a questioning mind intent upon gaining a thorough understanding of it.

A passive approach is doomed to failure. A pencil and paper should be at hand before you start reading. Of course, this means that you won't be able to read 20 pages a night; 3 pages would be a more reasonable goal, especially further along in the book where the level of abstraction is somewhat higher and more is expected of you. But as in anything where a considerable effort is required, the rewards are equally great; the satisfaction of writing a proof which you *know* is correct is hard to match. So pick up your pencil (or pen or whatever it is you use) and proceed at a deliberate pace through the following pages, knowing that mastery of their contents will lead to mathematical pleasures unknown to the uninitiated.

PREFACE

One of the most difficult steps a student of mathematics must make is the one into that (blissful) state known as "mathematical maturity." This is a step which is accomplished by making the transition from solving problems in a fairly concrete setting in which there is a well-known method or an algorithm for each problem type (as in most calculus courses, for example) to writing proofs and producing counterexamples involving more abstract objects and concepts, an activity for which there is no well-defined algorithm. Often this transition is something which is expected just to "happen," perhaps during the summer between the sophomore and junior years; however, it is not clear what summertime activities one could recommend to ensure such a result. My recent teaching experience suggests that this transition is not an easy one for most students and generally cannot be successfully made without some concerted effort and guidance. Two things which seem to inhibit a smooth transition are a lack of knowledge of some fundamental mathematical ideas—logic, sets, functions—and a lack of experience in two important mathematical activities—finding examples of objects with specified properties and writing proofs. This book is an attempt to provide an opportunity to gain exposure to these activities while learning some of the necessary fundamental ideas.

I have tried to keep the book as short as possible to achieve these goals; thus some interesting topics are left out and others are treated only in the exercises. I have also tried to take a developmental point of view so that the book starts out in a fairly simple, informal manner and gradually becomes more formal and abstract. This means that while it is possible to cover the first chapter rather rapidly, one should not expect to maintain this speed throughout the book; indeed, I have found that some sections in Chapter 2 can easily take more than a week to cover with any degree of thoroughness.

The transitional process begins with an informal introduction to logic, including a careful consideration of quantifiers and a discussion of basic

CHAPTER

1

LOGIC

1.1 INTRODUCTION

A friend of mine recently remarked that when he studied logic he got sleepy. I replied that he looked sleepy at the moment and he said, "Yes, I am sleepy." He added, "Therefore, you can conclude that I have been studying logic." "Most certainly not!" I answered. "That's a good example of an invalid argument. In fact, if you have been studying logic it's obvious that you haven't learned very much."

This short excerpt from a real-life situation is meant to illustrate the fact that we use logic in our everyday lives—although we don't always use it correctly. Logic provides the means by which we reach conclusions and establish arguments. Logic also provides the rules by which we reason in mathematics, and to be successful in mathematics we will need to understand precisely the rules of logic. Of course, we can also apply these rules to areas of life other than mathematics and amaze (or dismay) our friends with our logical, well-trained minds.

In this chapter we will describe the various connectives used in logic, develop some symbolic notation, discover some useful rules of inference, discuss quantification and display some typical forms of proof. Although our discussion of connectives and truth tables in the beginning is rather mechanistic and does not require much thought, by the end of the chapter we will be analyzing proofs and writing some of our own, a very non-mechanistic and thoughtful process.

It should be noted here that the truth table above does not have anything to do with p and q; they are just placeholders—cast in the same role as x in the familiar functional notation $f(x) = 2x - 3$. What the truth table does tell us, for example, is that when the first proposition is F and the second is T (third row of the table) the conjunction of the two propositions is F. You can check your understanding of this point by working exercise 5 at the end of this section.

Another common connective is "or," sometimes called *disjunction*. The disjunction of p and q, denoted by

$$p \vee q$$

is true when *at least one* of p, q is true. This is called the "inclusive or"; it corresponds to the "and/or" sometimes found in legal documents. Note that in ordinary conversation we often use "or" in the exclusive sense; true only when *exactly one* of the subpropositions is true. For example, the truth of "When you telephoned I must have been in the shower or walking the dog" isn't usually meant to include both possibilities. In mathematics we always use "or" in the inclusive sense as defined above and given in the truth table below:

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Given any proposition p we can form a new proposition with the opposite truth value, called the *negation* of p, which is denoted by

$$\neg p.$$

This is sometimes read as "not p."

The truth table for negation is

p	$\neg p$
T	F
F	T

1.2 AND, OR, NOT, AND TRUTH TABLES

The basic building blocks of logic are *propositions*. By a proposition we will mean a declarative sentence which is either true or false but not both. For example, "2 is greater than 3" and "All equilateral triangles are equiangular" are propositions while "$x < 3$" and "This sentence is false" are not (the first of these is a declarative sentence but we cannot assign a truth value until we know what "x" represents; try assigning a truth value to the second). We will denote propositions by lowercase letters, p, q, r, s, etc. In any given discussion different letters may or may not represent different propositions but a letter appearing more than once in a given discussion will always represent the same proposition. A true proposition will be given a truth value of T (for true) and a false proposition a truth value of F (for false). Thus "$2 + 3 < 7$" has a truth value of T while "$2 + 3 = 7$" has a truth value of F.

We are interested in combining simple propositions (sometimes called *subpropositions*) to make more complicated (or compound) propositions. We combine propositions with *connectives*, among which are "and," "or" and "implies." If p, q are two propositions then "p and q" is also a proposition, called the *conjunction* of p and q, and denoted by

$$p \wedge q.$$

The truth value of $p \wedge q$ depends on the truth values of the propositions p and q: $p \wedge q$ is true when p and q are both true, otherwise it is false. Notice that this is the usual meaning we assign to "and." The word "but" has the same logical meaning as "and" even though in ordinary English it carries a slightly different connotation. A convenient way to display this fact is by a *truth table*. As each of the two propositions p, q has two possible truth values, together they have $2 \times 2 = 4$ possible truth values so the table below lists all possibilities:

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Thus, for example, when p is T and q is F (line 2 of the truth table), $p \wedge q$ is F. In fact, this truth table can be taken as the definition of the connective $\wedge$.

We can form the negation of a proposition without understanding the meaning of the proposition by prefacing it with "it is false that" or "it is not the case that" but the resulting propositions are usually awkward and do not convey the real nature of the negation. A closer consideration of the meaning of the proposition in question will often indicate a better way of expressing the negation; later we will consider methods for negating compound propositions.

Consider the examples below:

a) $3 + 5 > 7$.
b) It is not the case that $3 + 5 > 7$.
c) $3 + 5 \leq 7$.
d) $x^2 - 3x + 2 = 0$ is not a quadratic equation.
e) It is not true that $x^2 - 3x + 2 = 0$ is not a quadratic equation.
f) $x^2 - 3x + 2 = 0$ is a quadratic equation.

Note that b) and c) are negations of a); e) and f) are negations of d), but c) and f) are to be preferred over b) and e), respectively.

We will use the same convention for $\neg$ as we use for $-$ in algebra; that is, it applies only to the next symbol, which in our case represents a proposition. Thus $\neg p \vee q$ will mean $(\neg p) \vee q$ rather than $\neg(p \vee q)$, just as $-3 + 4$ represents 1 and not -7. With this convention we can be unambiguous when we negate compound propositions using symbols, but life is not so easy when we consider how to negate compound propositions in English. For example, how do we distinguish between $\neg p \vee q$ and $\neg(p \vee q)$ in English? Suppose p represents "$2 + 2 = 4$," and q represents "$3 + 2 < 4$." Should "It is not the case that $2 + 2 = 4$ or $3 + 2 < 4$" mean $\neg(p \vee q)$ or $\neg p \vee q$? If we use the same convention we used for our symbols it should mean $\neg p \vee q$. But, if we take this meaning, then how would we say $\neg(p \vee q)$? The problem seems to be a lack of the equivalent of the parentheses we used for grouping. Let us adopt the convention that "it is not the case that" (or a similar negating phrase) applies to everything that follows, up to some sort of grouping punctuation. Thus, "It is not the case that $2 + 2 = 4$ or $3 + 2 < 4$" would mean $\neg(p \vee q)$, while "It is not the case that $2 + 2 = 4$, or $3 + 2 < 4$" would mean $\neg p \vee q$. Of course, when speaking, one must be very careful about using pauses to indicate the proper meaning.

Truth tables can be used to express the possible truth values of compound propositions by constructing the various columns in a methodical manner. For example, suppose that we wish to construct the truth table for $\neg(p \vee \neg q)$. We begin by making a basic four-row (there are four possibilities) truth table with column headings:

p	q	$\neg$	(	p	$\vee$	$\neg$	q	)
T	T							
T	F							
F	T							
F	F							

Truth values are then entered step by step:

p	q	$\neg$	(	p	$\vee$	$\neg$	q	)
T	T			T			T	
T	F			T			F	
F	T			F			T	
F	F			F			F	

p, q columns entered

p	q	$\neg$	(	p	$\vee$	$\neg$	q	)
T	T			T		F	T	
T	F			T		T	F	
F	T			F		F	T	
F	F			F		T	F	

$\neg q$ column entered

p	q	$\neg$	(	p	$\vee$	$\neg$	q	)
T	T			T	T	F	T	
T	F			T	T	T	F	
F	T			F	F	F	T	
F	F			F	T	T	F	

$p \vee \neg q$ column entered

p	q	$\neg$	(	p	$\vee$	$\neg$	q	)
T	T	**F**		T	T	F	T	
T	F	**F**		T	T	T	F	
F	T	**T**		F	F	F	T	
F	F	**F**		F	T	T	F	

$\neg(p \vee \neg q)$ column entered

After some experience is obtained, many of the steps written above can be eliminated. We also note that if a compound proposition involves n subpropositions then its truth table will require 2^n rows. Thus a compound proposition with four subpropositions will require $2^4 = 16$ rows.

Exercises 1.2

1. Assign truth values to the following propositions:
  a) $3 \leq 7$ and 4 is an odd integer.
  b) $3 \leq 7$ or 4 is an odd integer.
  c) $2 + 1 = 3$ but $4 < 4$.
  d) 5 is odd or divisible by 4.
  e) It is not true that $2 + 2 = 5$ and $5 > 7$.
  f) It is not true that $2 + 2 = 5$ or $5 > 7$.
  g) $3 \geq 3$.

2. Suppose that we represent "7 is an even integer" by p, "$3 + 1 = 4$" by q and "24 is divisible by 8" by r.
  a) Write the following in symbolic form and assign truth values:
    i) $3 + 1 \neq 4$ and 24 is divisible by 8.
    ii) It is not true that 7 is odd or $3 + 1 = 4$.
    iii) $3 + 1 = 4$ but 24 is not divisible by 8.
  b) Write out the following in words and assign truth values:
    i) $p \vee \neg q$.
    ii) $\neg(r \wedge q)$.
    iii) $\neg r \vee \neg q$.

3. Construct truth tables for
  a) $\neg p \vee q$.
  b) $\neg p \wedge p$.
  c) $(\neg p \vee q) \wedge r$.
  d) $\neg(p \wedge q)$.
  e) $\neg p \wedge \neg q$.
  f) $\neg p \vee \neg q$.

g) $p \vee \neg p$.
h) $\neg(\neg p)$.

4. Give useful negations of
a) $3 - 4 < 7$.
b) $3 + 1 = 5$ and $2 \leq 4$.
c) 8 is divisible by 3 but 4 is not.

5. Suppose that we define the connective $\star$ by saying that $p \star q$ is true only when q is true and p is false and is false otherwise.
a) Write out the truth table for $p \star q$.
b) Write out the truth table for $q \star p$.
c) Write out the truth table for $(p \star p) \star q$.

6. Let us denote the "exclusive or" sometimes used in ordinary conversation by $\oplus$. Thus $p \oplus q$ will be true when exactly one of p, q is true, and false otherwise.
a) Write out the truth table for $p \oplus q$.
b) Write out the truth tables for $p \oplus p$ and $(p \oplus q) \oplus q$.
c) Show that "and/or" really means "and or or," that is, the truth table for $(p \wedge q) \oplus (p \oplus q)$ is the same as the truth table for $p \vee q$.
d) Show that it makes no difference if we take both "or's" in "and/or" to be inclusive ($\vee$) or exclusive ($\oplus$).

1.3 IMPLICATION AND THE BICONDITIONAL

If we were to write out the truth tables for $\neg(p \wedge q)$ and for $\neg p \vee \neg q$ (as we did in exercise 3 d), f) above) and compare them, we would note that these two propositions have the same truth values and thus in some sense are the same. This is an important concept (important enough to have a name anyway) so we make the following definition:

Suppose that two propositions p, q have the same truth table. Then p and q are said to be *logically equivalent*, which we will denote by

$$p \Longleftrightarrow q.$$

Basically, when two propositions are logically equivalent they have the same form and we may substitute one for the other in any other proposition or theorem. It is important to emphasize that it is the *form* and not the truth value of a proposition which determines whether it is (or is not) logically equivalent to another proposition. For example, "$2 + 2 = 4$" and "$7 - 5 = 2$" are both true propositions but they are *not* logically equivalent since they have different truth tables (if we represented the first by p then the other would need another symbol, say q, and we know that these do not have the same truth tables). On the other hand, "2 + 3 = 5 or 3 − 4 = 2" and "3 − 4 = 2 or 2 + 3 = 5" *are* logically equivalent. To see this, let p represent "3 − 4 = 2" and q represent "2 + 3 = 5." Then the first is of the form $q \vee p$ while the second has the form $p \vee q$. A check of truth tables shows that these two do indeed have the same truth table.

Using this idea of logical equivalence we can state the relationship between negation, disjunction and conjunction, sometimes called *DeMorgan's laws*:

Let p, q be any propositions. Then

$$\neg(p \vee q) \Longleftrightarrow \neg p \wedge \neg q,$$

$$\neg(p \wedge q) \Longleftrightarrow \neg p \vee \neg q.$$

We have already verified the second of these in exercise 3 d) and f) in the previous section; the reader should verify the other by means of a truth table now. In words, DeMorgan's laws state that the negation of a disjunction is logically equivalent to the conjunction of the negations and the negation of a conjunction is logically equivalent to the disjunction of the negations. A common mistake is to treat $\neg$ in logic as $-$ in algebra and to think that $\neg$ distributes over $\vee$ and $\wedge$ just as $-$ distributes over $+$. That is, since $-(a + b) = -a + (-b)$, one might be led to believe, for example, that $\neg(p \vee q) \Leftrightarrow \neg p \vee \neg q$. A quick check with truth tables (or reference to exercise 3 d), e) in the previous section) shows that this isn't correct. Thus, while our logical notation appears somewhat "algebra-like" (and indeed is an example of a certain kind of algebra) its rules differ from those of the familiar algebra of real numbers and we should not make the mistake of assuming that certain logical operations behave in ways analogous to our old algebraic friends $+$, $\times$ and $-$.

One of the most important propositional forms in mathematics is that of implication, sometimes called the *conditional*. In fact, all mathematical theorems are in the form of an implication: If "hypothesis" then "conclusion." The general form of an implication is "if p then q" where p, q are propositions; we will denote this by

$$p \rightarrow q.$$

In the conditional $p \rightarrow q$, p is called the *premise* (or hypothesis or antecedent) and q is called the *conclusion* (or *consequence* or *consequent*). The truth table for $p \rightarrow q$ is

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

If we think about the usual meaning we give to *implies* we should agree that the first two lines of the above truth table correspond to ordinary usage, but that the last two lines may not be so clear. Of course, we are free to define the truth values of the various connectives in any way we choose and we could take the position that this is the way we want to define *implies* (which is indeed the case) but it is worthwhile to see that the definition above also agrees with everyday usage. To this end, let us consider what might be called "The parable of the dissatisfied customer." Imagine that we have purchased a product, say a washday detergent called *Tyde*, after hearing an advertisement which said, "If you use Tyde then your wash will be white!" Under what circumstances could we complain to the manufacturer? A little thought reveals that we certainly couldn't complain if we had not used Tyde (the ad said nothing about what would happen if we used Chear, for example), and we couldn't complain if we used Tyde and our wash was white; thus we could complain only in the case when we had used Tyde and our wash was not white (as promised). Thus, the ad's promise is false only when "we use Tyde and get a non-white wash" is true. Let's use our logic notation to examine this situation more closely. Let p represent "We use Tyde," and q represent "Our wash is white." Then the advertisement's promise is

$$p \to q$$

and we can complain (that is, this promise is false) only in the case when

$$p \wedge \neg q$$

is true. Thus, $p \wedge \neg q$ should be logically equivalent to $\neg(p \to q)$. Writing out the truth table for $p \wedge \neg q$ we get (the reader should verify this):

p	q	$p \wedge \neg q$
T	T	F
T	F	T
F	T	F
F	F	F

As this is to be logically equivalent to the negation of $p \to q$, the truth table for $p \to q$ should be the negation of this (which it is—look back to check this) and our logic definition of implication does agree with our everyday (or at least washday) usage.

We note that the only case in which $p \to q$ is false is when p is true and q is false; that is, when the hypothesis is true and the conclusion is false. Thus the following implications are all true:

a) If $2 + 2 = 4$ then $1 + 1 = 2$.
b) If $2 + 3 = 4$ then $1 + 1 = 5$.
c) If green is red then the moon is made of cheese.
d) If green is red then the moon is not made of cheese.
e) $7 < 2$ if $2 < 1$.

It should also be noted that if an implication is true then its conclusion may be true or false (see examples a), b) above), but if an implication is true *and* the hypothesis is true then the conclusion *must* be true. This, of course, is the basic form of a mathematical theorem: if we know the theorem (an implication) is correct (true) and the hypothesis of the theorem is true we can take the conclusion of the theorem to be true.

There are many ways of stating the conditional in English and all the following are considered logically equivalent:

a) If p then q.
b) p implies q.
c) p is stronger than q.
d) q is weaker than p.
e) p only if q.
f) q if p.
g) p is sufficient for q.
h) q is necessary for p.

i) A necessary condition for p is q.
j) A sufficient condition for q is p.

Most of the time we will use the first two, but it is important to be familiar with the rest. Keeping in mind the definition of $p \rightarrow q$ will help us to remember some of these. For example, when we say "r is sufficient for s," we mean that the truth of r is sufficient to guarantee the truth of s; that is, we mean $r \rightarrow s$. Similarly, if we say "r is necessary for s," we mean that when s is true, r must necessarily be true too; that is, we mean $s \rightarrow r$.

When we observe the truth table for $p \rightarrow q$ we note that it is not symmetric with respect to p and q; that is, the truth table for $p \rightarrow q$ is not the same as the truth table for $q \rightarrow p$. In other words, these two propositions are *not* logically equivalent and thus cannot be substituted one for another. Because of this lack of symmetry it is convenient to make the following definitions:

Given an implication $p \rightarrow q$:

$q \rightarrow p$ is called its *converse*,
$\neg q \rightarrow \neg p$ is called its *contrapositive*,
$\neg p \rightarrow \neg q$ is called its *inverse*.

Even though the reader has probably already noticed it, it is worth pointing out that the inverse of an implication is the contrapositive of its converse (it is also the converse of its contrapositive).

Perhaps the most common logical error is that of confusing an implication with its converse (or inverse). In fact, this common error seems to be the basis for much advertising. For example, if we are told that "If we use Tyde then our wash will be white!" (which may be true) we are apparently expected to also believe that if we don't use Tyde then our wash won't be white. But this is the inverse, which is logically equivalent to the converse, of the original claim. Thus, we see that we can believe Tyde's claim and still use Chear with a clear (logically, anyway) conscience and wear white clothes. However, an implication and its contrapositive *are* logically equivalent (see exercises below) and thus may be used interchangeably. In this case, this means that if our clothes are not white then we didn't use Tyde.

The final connective which we will consider is the *biconditional*. If p, q are two propositions then the proposition "p if and only if q" (sometimes abbreviated "p iff q"), denoted by

$$p \leftrightarrow q,$$

is called the *biconditional* (not to be confused with logical equivalence "$\Leftrightarrow$," although there is a connection which will be revealed in the next section; keep reading). We say that $p \leftrightarrow q$ is true when p, q have the same truth value and false when they have different truth values. Thus the truth table for the biconditional is

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Some other ways of expressing $p \leftrightarrow q$ are

> p is necessary and sufficient for q.
> p is equivalent to q.

As the names (biconditional, if and only if) and notation suggest, there is a close connection between the conditional and the biconditional. In fact, $p \leftrightarrow q$ is logically equivalent to $(p \rightarrow q) \wedge (q \rightarrow p)$.

Exercises 1.3

1. Which of the following are logically equivalent?
- a) $p \wedge \neg q$.
- b) $p \rightarrow q$.
- c) $\neg(\neg p \vee q)$.
- d) $q \rightarrow \neg p$.
- e) $\neg p \vee q$.
- f) $\neg(p \rightarrow q)$.
- g) $p \rightarrow \neg q$.
- h) $\neg p \rightarrow \neg q$.

2. Show that the following pairs are logically equivalent:
- a) $p \wedge (q \vee r)$; $(p \wedge q) \vee (p \wedge r)$.
- b) $p \vee (q \wedge r)$; $(p \vee q) \wedge (p \vee r)$.
- c) $p \leftrightarrow q$; $(p \rightarrow q) \wedge (q \rightarrow p)$.
- d) $p \rightarrow q$; $\neg q \rightarrow \neg p$.

3. Show that the following pairs are not logically equivalent:

a) $\neg(p \wedge q)$; $\neg p \wedge \neg q$.

b) $\neg(p \vee q)$; $\neg p \vee \neg q$.

c) $p \rightarrow q$; $q \rightarrow p$.

d) $\neg(p \rightarrow q)$; $\neg p \rightarrow \neg q$.

4. Find:

a) The contrapositive of $\neg p \rightarrow q$.

b) The converse of $\neg q \rightarrow p$.

c) The inverse of the converse of $q \rightarrow \neg p$.

d) The negation of $p \rightarrow \neg q$.

e) The converse of $\neg p \wedge q$.

5. Indicate which of the following is true:

a) If $2 + 1 = 4$ then $3 + 2 = 5$.

b) Red is white if and only if green is blue.

c) $2 + 1 = 3$ and $3 + 1 = 5$ implies 4 is odd.

d) If 4 is odd then 5 is odd.

e) If 4 is odd then 5 is even.

f) If 5 is odd then 4 is odd.

6. Give examples of or tell why no such example exists:

a) A true implication with a false conclusion.

b) A true implication with a true conclusion.

c) A false implication with a true conclusion.

d) A false implication with a false conclusion.

e) A false implication with a false hypothesis.

f) A false implication with a true hypothesis.

g) A true implication with a true hypothesis.

h) A true implication with a false hypothesis.

7. Translate into symbols:

a) p whenever q.

b) p unless q.

8. Give a negation for $p \leftrightarrow q$ in a form which does not involve a biconditional.

9. Suppose that p, $\neg q$ and r are true. Which of the following is true?

a) $p \rightarrow q$.

b) $q \rightarrow p$.

c) $p \rightarrow (q \vee r)$.

d) $p \leftrightarrow q$.

e) $p \leftrightarrow r$.

f) $(p \vee q) \rightarrow p$.

g) $(p \wedge q) \rightarrow q$.

10. We note that we now have five logic "connectives": $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$ and $\neg$, each of which corresponds to a construct from our ordinary language. It turns out that from a logical point of view this is somewhat wasteful, since we could express all these in terms of just $\neg$ and $\wedge$. Even more, if we define $p \mid q$ to be false when both p and q are true and true otherwise, we could express all five forms in terms of this one connective ($\mid$ is known as the Sheffer stroke). Partially verify the statements given above by

a) Finding a proposition which is equivalent to $p \vee q$ using just $\wedge$ and $\neg$.
b) Writing out the truth table for $p \mid q$.
c) Showing that $p \mid p$ is equivalent to $\neg p$.
d) Showing that $(p \mid q) \mid (q \mid p)$ is equivalent to $p \wedge q$.

1.4 TAUTOLOGIES

An important class of propositions are those whose truth tables contain only T's in the final column; that is, propositions which are always true and the fact that they are always true depends only on their form and not on any meaning which might be assigned to them (for example, recall exercise 3 g) of section 1.2: $p \vee \neg p$). Such propositions are called *tautologies*. It is important to distinguish between true propositions and tautologies. For example, "2 + 2 = 4" is a true proposition but it is not a tautology because its form is p which is not always true. On the other hand, "5 is a primitive root of 17 or 5 is not a primitive root of 17" is a tautology no matter what being a primitive root means. It is a tautology by virtue of its form $(p \vee \neg p)$ alone.

The negation of a tautology, that is, a proposition which is always false, is called a *contradiction*. We must distinguish between contradictions and false statements in the same way we distinguish between true statements and tautologies; a proposition is a contradiction based on its form alone. As examples, consider the truth tables:

p	q	p	$\rightarrow$	$(p$	$\vee$	$q)$
T	T	T	**T**	T	T	T
T	F	T	**T**	T	T	F
F	T	F	**T**	F	T	T
F	F	F	**T**	F	F	F

p	q	$(p$	$\rightarrow$	$q)$	$\wedge$	$(p$	$\wedge$	$\neg q)$
T	T	T	T	T	**F**	T	F	F
T	F	T	F	F	**F**	T	T	T
F	T	F	T	T	**F**	F	F	F
F	F	F	T	F	**F**	F	F	T

We see that $p \rightarrow (p \vee q)$ is a tautology and $(p \rightarrow q) \wedge (p \wedge \neg q)$ is a contradiction.

Using the idea of tautology, perhaps we can make clear the distinction between "equivalent" and "logically equivalent." Two propositions p, q are logically equivalent if and only if $p \leftrightarrow q$ is a tautology. Actually, $p \leftrightarrow q$ and $p \Longleftrightarrow q$ are propositions on two different levels. If we think of "p is equivalent to q" as a proposition, then "p is logically equivalent to q" is a proposition about this proposition; namely, the (meta)-proposition "p is equivalent to q is true." For example, $(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$ is a logical implication while $p \rightarrow (p \wedge q)$ is not; it is "just" an implication which may or may not be true.

We also use the idea of tautology to make the following definition: we say that $p \rightarrow q$ is a *logical implication* (also "p logically implies q" or "q is a logical consequence of p") if $p \rightarrow q$ is a tautology. p logically implies q is denoted by

$$p \Rightarrow q.$$

Note that logical implication bears the same relation to implication as logical equivalence bears to equivalence. If p logically implies q, and p is true, then q must also be true. For example, $p \rightarrow (p \vee q)$, $(p \wedge q) \rightarrow p$ are logical implications while $p \rightarrow (p \wedge q)$ is not (when p is T and q is F this last implication is F and hence not a tautology).

Tautologies form the rules by which we reason and for future reference a list of the more common ones, along with some of their names, is given below (p, q, r represent any propositions, $\mathbf{c}$ represents any contradiction, $\mathbf{t}$ represents any tautology).

A list of tautologies

1.	$p \vee \neg p$	
2.	$\neg(p \wedge \neg p)$	
3.	$p \rightarrow p$	
4.	a) $p \leftrightarrow (p \vee p)$ b) $p \leftrightarrow (p \wedge p)$	idempotent laws
5.	$\neg\neg p \leftrightarrow p$	double negation
6.	a) $(p \vee q) \leftrightarrow (q \vee p)$ b) $(p \wedge q) \leftrightarrow (q \wedge p)$ c) $(p \leftrightarrow q) \leftrightarrow (q \leftrightarrow p)$	commutative laws
7.	a) $(p \vee (q \vee r)) \leftrightarrow ((p \vee q) \vee r)$ b) $(p \wedge (q \wedge r)) \leftrightarrow ((p \wedge q) \wedge r)$	associative laws
8.	a) $(p \wedge (q \vee r)) \leftrightarrow ((p \wedge q) \vee (p \wedge r))$ b) $(p \vee (q \wedge r)) \leftrightarrow ((p \vee q) \wedge (p \vee r))$	distributive laws
9.	a) $(p \vee \mathbf{c}) \leftrightarrow p$ b) $(p \wedge \mathbf{c}) \leftrightarrow \mathbf{c}$ c) $(p \vee \mathbf{t}) \leftrightarrow \mathbf{t}$ d) $(p \wedge \mathbf{t}) \leftrightarrow p$	identity laws
10.	a) $\neg(p \wedge q) \leftrightarrow (\neg p \vee \neg q)$ b) $\neg(p \vee q) \leftrightarrow (\neg p \wedge \neg q)$	DeMorgan's laws
11.	a) $(p \leftrightarrow q) \leftrightarrow ((p \rightarrow q) \wedge (q \rightarrow p))$ b) $(p \leftrightarrow q) \leftrightarrow ((p \wedge q) \vee (\neg p \wedge \neg q))$ c) $(p \leftrightarrow q) \leftrightarrow (\neg p \leftrightarrow \neg q)$	equivalance
12.	a) $(p \rightarrow q) \leftrightarrow (\neg p \vee q)$ b) $\neg(p \rightarrow q) \leftrightarrow (p \wedge \neg q)$	implication
13.	$(p \rightarrow q) \leftrightarrow (\neg q \rightarrow \neg p)$	contrapositive
14.	$(p \rightarrow q) \leftrightarrow ((p \wedge \neg q) \rightarrow \mathbf{c})$	reductio ad absurdum
15.	a) $((p \rightarrow r) \wedge (q \rightarrow r)) \leftrightarrow ((p \vee q) \rightarrow r))$ b) $((p \rightarrow q) \wedge (p \rightarrow r)) \leftrightarrow (p \rightarrow (q \wedge r))$	
16.	$((p \wedge q) \rightarrow r) \leftrightarrow (p \rightarrow (q \rightarrow r))$	exportation law
17.	$p \rightarrow (p \vee q)$	addition
18.	$(p \wedge q) \rightarrow p$	simplification
19.	$(p \wedge (p \rightarrow q)) \rightarrow q$	modus ponens
20.	$((p \rightarrow q) \wedge \neg q) \rightarrow \neg p$	modus tollens
21.	$((p \rightarrow q) \wedge (q \rightarrow r)) \rightarrow (p \rightarrow r)$	hypothetical syllogism
22.	$((p \vee q) \wedge \neg p) \rightarrow q$	disjunctive syllogism
23.	$(p \rightarrow \mathbf{c}) \rightarrow \neg p$	absurdity
24.	$((p \rightarrow q) \wedge (r \rightarrow s)) \rightarrow ((p \vee r) \rightarrow (q \vee s))$	
25.	$(p \rightarrow q) \rightarrow ((p \vee r) \rightarrow (q \vee r))$	

Observe that in the above list, 4-16 are logical equivalences while 17-25 are logical implications.

One of the first questions from students when they see the above list is, "Do we have to memorize all of these?" The answer is, "No, memorization is not sufficient, you need to *know* all these! They need to be incorporated into your way of thinking." At first glance, this may seem like a formidable task, and perhaps it is. But some of these are already incorporated into our way of thinking. For example, if someone says, "This sweater is orlon or wool. It isn't orlon," what do we conclude about the sweater? We conclude that it is a wool sweater, and in doing so we have just used the disjunctive syllogism (22 on the list above). Similarly, if someone says, "If I do the assignments then I enjoy the class. I did the assignment for today," we conclude that the person speaking enjoyed the class today. This is an application of the modus ponens (19 on the list). It is not important that we learn the names of the various equivalences and implications, but it is important that we learn their *forms* so that we can recognize when we are using them. It is also important to recognize when we are not reasoning correctly; that is, when we use something which is not a logical implication. In the next section we will spend some time looking at this point.

Exercises 1.4

1. Verify that 7 a), 9 b), 13 and 14 in the list above are tautologies.

2. Determine which of the following have the form of something on the above list (for example, $(\neg q \wedge p) \rightarrow \neg q$ has the form of 18) and in these cases, indicate which one:

a) $\neg q \rightarrow (\neg q \vee \neg p)$.
b) $q \rightarrow (q \wedge \neg p)$.
c) $(r \rightarrow \neg p) \leftrightarrow (\neg r \vee \neg p)$.
d) $(p \rightarrow \neg q) \leftrightarrow \neg(\neg p \rightarrow q)$.
e) $(\neg r \rightarrow q) \leftrightarrow (\neg q \rightarrow r)$.
f) $(p \rightarrow (\neg r \vee q)) \leftrightarrow ((r \wedge \neg q) \rightarrow \neg p)$.
g) $r \rightarrow \neg(q \wedge \neg r)$.
h) $((\neg q \vee p) \wedge q) \rightarrow p$.

3. Give examples of or tell why no such example exists:

a) A logical implication with a false conclusion.
b) A logical implication with a true conclusion.
c) A logical implication with a true hypothesis and false conclusion.

4. Which of the following are correct?

a) $(p \rightarrow (q \vee r)) \Rightarrow (p \rightarrow q)$.
b) $((p \vee q) \rightarrow r) \Rightarrow (p \rightarrow r)$.
c) $(p \vee (p \wedge q)) \Leftrightarrow p$.
d) $((p \rightarrow q) \wedge \neg p) \Rightarrow \neg q$.

5. Which of the following are tautologies, contradictions or neither?
   a) $(p \wedge \neg q) \rightarrow (q \vee \neg p)$.
   b) $\neg p \rightarrow p$.
   c) $\neg p \leftrightarrow p$.
   d) $(p \wedge \neg p) \rightarrow p$.
   e) $(p \wedge \neg p) \rightarrow q$.
   f) $(p \wedge \neg q) \leftrightarrow (p \rightarrow q)$.
   g) $[(p \rightarrow q) \leftrightarrow r] \leftrightarrow [p \rightarrow (q \leftrightarrow r)]$.
6. Which of the following are correct?
   a) $(p \leftrightarrow q) \Rightarrow (p \rightarrow q)$.
   b) $(p \rightarrow q) \Rightarrow (p \leftrightarrow q)$.
   c) $(p \rightarrow q) \Rightarrow q$.
7. Is $\rightarrow$ associative; i.e., is $((p \rightarrow q) \rightarrow r) \Longleftrightarrow (p \rightarrow (q \rightarrow r))$?
8. Is $\leftrightarrow$ associative; i.e., is $((p \leftrightarrow q) \leftrightarrow r) \Longleftrightarrow (p \leftrightarrow (q \leftrightarrow r))$?
9. Which of the following true propositions are tautologies?
   a) If $2 + 2 = 4$ then 5 is odd.
   b) $3 + 1 = 4$ and $5 + 3 = 8$ implies $3 + 1 = 4$.
   c) $3 + 1 = 4$ and $5 + 3 = 8$ implies $3 + 2 = 5$.
   d) Red is yellow or red is not yellow.
   e) Red is yellow or red is red.
   f) 4 is odd or 2 is even and 2 is odd implies 4 is odd.
   g) 4 is odd or 2 is even and 2 is odd implies 4 is even.
10. Which of the following are logical consequences of the set of propositions $p \vee q$, $r \rightarrow \neg q$, $\neg p$?
    a) q.
    b) r.
    c) $\neg p \vee s$.
    d) $\neg r$.
    e) $\neg(\neg q \wedge r)$.
    f) $q \rightarrow r$.

1.5 ARGUMENTS AND THE PRINCIPLE OF DEMONSTRATION

How do you win an argument? Aside from intimidation, force of personality, coercion or threats, of course; we are speaking of convincing someone of the logical correctness of your position. You might begin by saying, "Do you accept p, q and r as being true?" If the answer is, "Yes, of course. Any dolt can see that!" then you say, "Well then, it follows that t must be true." For you to win your argument it must be the case (and this is what you must argue) that $(p \wedge q \wedge r) \rightarrow t$ is a tautology; that is, there is no