

5.2: Gram-Schmidt Orthogonalization and QR Factorization

Concept: use a given basis $\vec{v}_1, \dots, \vec{v}_m$ to construct an orthonormal basis $\vec{u}_1, \dots, \vec{u}_m$

ex 1: $\vec{v}_1 = \begin{bmatrix} 2 \\ 2 \\ -2 \\ 2 \end{bmatrix} \Rightarrow \vec{u}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$

$\|\vec{v}_1\| = 4$ check: $\vec{v}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{bmatrix} [4]$

ex 2: $\vec{v}_1 = \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}$ & $\vec{v}_2 = \begin{bmatrix} 25 \\ 0 \\ -25 \end{bmatrix}$

$\|\vec{v}_1\| = 5$

$\vec{u}_1 = \begin{bmatrix} 4/5 \\ 0 \\ 3/5 \end{bmatrix}$

$\vec{v}_2^\perp = \vec{v}_2 - \underbrace{(\vec{u}_1 \cdot \vec{v}_2)}_5 \vec{u}_1 = \begin{bmatrix} 25 \\ 0 \\ -25 \end{bmatrix} - \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 21 \\ 0 \\ -28 \end{bmatrix}$

and $\|\vec{v}_2^\perp\| = 35$ so $\vec{u}_2 = \begin{bmatrix} 3/5 \\ 0 \\ -4/5 \end{bmatrix}$

check: $\begin{bmatrix} 4 & 25 \\ 0 & 0 \\ 3 & -25 \end{bmatrix} = \begin{bmatrix} 4/5 & 3/5 \\ 0 & 0 \\ 3/5 & -4/5 \end{bmatrix} \begin{bmatrix} 5 & 5 \\ 0 & 35 \end{bmatrix}$

ex 3: $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix}$

$\|\vec{v}_1\| = 2$

$\vec{u}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$

$\vec{v}_2^\perp = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - (u_1 \cdot v_2) \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$

and $\|\vec{v}_2^\perp\| = 1$ so $\vec{u}_2 = \vec{v}_2^\perp$

$\vec{v}_3^\perp = \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix} - (u_1 \cdot v_3) \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} + (u_2 \cdot v_3) \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ +1/2 \\ -1/2 \\ -1/2 \end{bmatrix}$

and $\|\vec{v}_3^\perp\| = 1$ so $\vec{u}_3 = \vec{v}_3^\perp$

check: $\begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

\mathbb{Q} \mathbb{R}

ex4: Find the QR Factorization of the matrix M.

$$M = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ 1 & 1 & 0 \\ 1 & 0 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \|\vec{v}_1\| = 2 \quad \text{and} \quad \vec{u}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

$$\vec{v}_2^\perp = \vec{v}_2 - \vec{v}_2'' = \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2) \vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - (1) \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$$

$$\vec{u}_2 = \frac{\vec{v}_2^\perp}{\|\vec{v}_2^\perp\|} \leftarrow 1 = \vec{v}_2^\perp$$

$$\vec{v}_3^\perp = \vec{v}_3 - (\vec{u}_1 \cdot \vec{v}_3) \vec{u}_1 - (\vec{u}_2 \cdot \vec{v}_3) \vec{u}_2 = \vec{v}_3 - (1) \vec{u}_1 - (-2) \vec{u}_2 = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ -1/2 \end{bmatrix}$$

$$\vec{u}_3 = \frac{\vec{v}_3^\perp}{\|\vec{v}_3^\perp\|} \leftarrow 1$$

and so $M = QR$ where

$$Q = \begin{bmatrix} 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 \end{bmatrix}_{4 \times 3}$$

$$\text{and} \quad R = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}_{3 \times 3}$$