4.3: Matrix of a L.T.

ex: $T(f(t)) = f(2t-1)$ from P_2 to P_2

$$\left.\begin{aligned} T(1) &= 1 \\ T(t) &= 2t-1 \\ T(t^2) &= (2t-1)^2 = 4t^2-4t+1 \end{aligned}\right\} \text{L.I.}$$

T is an isomorphism. $\left(\text{im}(T) = P_2 \ \& \ \ker(T) = 0 \right)$

$$\begin{tikzcd}
a+bt+ct^2 \arrow[r, "T"] \arrow[d, "L_u"'] & a+b(2t-1)+c(4t^2-4t+1) \\
 & = (a-b+c)+(2b-4c)t+4ct^2 \\
\begin{bmatrix} a \\ b \\ c \end{bmatrix} \arrow[r, "B"] & \begin{bmatrix} a-b+c \\ 2b-4c \\ 4c \end{bmatrix} \arrow[u, "L_u^{-1}"']
\end{tikzcd}$$

$$B = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -4 \\ 0 & 0 & 4 \end{bmatrix}$$

<u>ex</u>: $T(z) = (p+iq)z$ from $\mathbb{C}$ to $\mathbb{C}$, p & $q \in \mathbb{R}$.

$$\left.\begin{array}{l} T(1) = (p+iq)1 = p+iq \\ T(i) = (p+iq)i = -q+pi \end{array}\right\} \text{L.I.}$$

T is an isomorphism. $\left(\operatorname{im}(T) = \mathbb{C} \text{ and } \ker(T) = 0 \right)$

$$\begin{tikzcd}
a+bi \arrow[r, "(p+iq)z", "T"'] \arrow[d, "L_{\mathcal{U}}"'] & (ap-bq)+i(aq+bp) \\
\begin{bmatrix} a \\ b \end{bmatrix}_{\mathcal{U}} \arrow[r, "B", "{\begin{bmatrix} p & -q \\ q & p \end{bmatrix}}"'] & \begin{bmatrix} ap-bq \\ aq+bp \end{bmatrix}_{\mathcal{U}} \arrow[u, "L_{\mathcal{U}}^{-1}"']
\end{tikzcd}$$

ex: $T(m) = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} m$ from $\mathbb{R}^{2\times 2}$ to $\mathbb{R}^{2\times 2}$,

$$T\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}$$

$$T\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$$

only two L.I. matrices so T is not an isomorphism

$$\operatorname{im}(T) = \operatorname{span}\left(\begin{bmatrix} 1 & 0 \\ 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}\right)$$

$$\ker(T) = \operatorname{span}\left(\begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}\right)$$

$$\begin{tikzcd}
\begin{bmatrix} a & b \\ c & d \end{bmatrix} \arrow[r, "A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}"] \arrow[d, "L_{\mathcal{U}}"'] & \begin{bmatrix} a+c & b+d \\ 2a+2c & 2b+2d \end{bmatrix} \arrow[d, "L_{\mathcal{U}}"] \\
\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}_{\mathcal{U}} \arrow[r] & \begin{bmatrix} a+c \\ b+d \\ 2a+2c \\ 2b+2d \end{bmatrix}_{\mathcal{U}}
\end{tikzcd}$$

$$B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix}$$