4.2 : Lin Trans and Isomorphisms

Def: Consider two lin. spaces V & W. $T: V \to W$ is a lin. trans. if

(i) $T(f+g) = T(f) + T(g)$

(ii) $T(kf) = kT(f)$

$\forall\ f, g \in V$ and scalars k

(i) $\text{im}(T) = \{T(f) \mid f \in V\}$

(ii) $\ker(T) = \{f \in V \mid T(f) = 0\}$

If the image of T is finite dim. then

$\dim(\text{im}\,T) = \text{rank}(T)$

If the ker of T is finite dim. then

$\dim(\ker T) = \text{nullity}(T)$

And if V is finite dim:

$$\dim(V) = \text{rank}(T) + \text{nullity}(T)$$
$$= \dim(\text{im}\,T) + \dim(\ker T)$$

ex1: Is the transformation linear?

$$T(M) = \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} M \quad \text{from } \mathbb{R}^2 \mapsto \mathbb{R}^2$$

ex2: Is the transformation linear?

$T(M) = PMQ$ where $P = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}$ & $Q = \begin{bmatrix} 3 & 5 \\ 7 & 11 \end{bmatrix}$

from $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$

Find the kernel & image in (ex1) & Ex2).

Dfn: An <u>invertible linear transformation</u> is called an isomorphism.

Coordinate Transformations are isomorphisms.

<u>Key</u>: Any n-dim. lin. space is isomorphic w/ $\mathbb{R}^n$.

Nice example: P_n is isomorphic w/ $\mathbb{R}^{n+1}$!

<u>Thm</u>: Properties of isomorphisms.

- $T: V \mapsto W$ is an isomorphism iff $\ker(T) = \{0\}$ and $\operatorname{im}(T) = W$.
- If V is isomorphic to W, then $\dim V = \dim W$
- If $T: V \mapsto W$ is a L.T. w/ $\ker(T) = \{0\}$ then T is an isomorphism.
- If $T: V \mapsto W$ is a L.T. w/ $\operatorname{im}(T) = W$. If $\dim V = \dim W$ then T is an isomorphism.

Are (ex1) & (ex2) isomorphisms?

ex 3: $T(f(t)) = f(7)$ from P_2 to $\mathbb{R}$.

LT

ker

im

isomorphism

ex4: $T(f(t)) = f'(t)$ from P_2 to P_2

LT

ker

im

isomorphism