7.4: Diagonalization

The Plan

T : 7.4 Diagonalization, T : 7.5

W : 7.4 W : 7.6 stability.

Th : 7.5 ℂ Th : 7.6

F : HW (7.1–4) F : HW (7.4–6)

Note: I need study group sign-ins.

Focus: Dynamical Systems.

Picture

Coyotes & Road Runners

Symbolic

$\vec{x}(t+1) = A\vec{x}(t)$ w/ $\vec{x}(0) = \vec{x}_0$

$\vec{x}(t)$ = state vector

$\vec{x}_0$ = initial state vector.

Graphically

phase Portrait

Ex1: Remember our coyote & roadrunner example from (7.1).

$$A = \begin{bmatrix} 0.86 & 0.08 \\ -0.12 & 1.14 \end{bmatrix}$$

$$E_{1.1} = \text{Span}\left(\begin{bmatrix} 100 \\ 300 \end{bmatrix} \right)$$

$$E_{0.9} = \text{span}\left(\begin{bmatrix} 200 \\ 100 \end{bmatrix} \right)$$

Goal: Find a closed form equation for $\vec{X}(t)$.

$$\begin{tikzcd}
\vec{X}(t) = c_1 \underbrace{\begin{bmatrix} 100 \\ 300 \end{bmatrix}}_{\vec{v}_1} + c_2 \underbrace{\begin{bmatrix} 200 \\ 100 \end{bmatrix}}_{\vec{v}_2} \arrow[r, "A"] \arrow[d, "S^{-1}"'] & \begin{aligned} T(\vec{X}) &= A\vec{x} \\ &= c_1 A\vec{v}_1 + c_2 A\vec{v}_2 \\ &= 1.1c_1\vec{v}_1 + 0.9c_2\vec{v}_2 \end{aligned} \\
\left[\vec{X}(t)\right]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \arrow[r, "D = \begin{bmatrix} 1.1 & 0 \\ 0 & 0.9 \end{bmatrix}"] & \left[T(\vec{X})\right]_{\mathcal{B}} = \begin{bmatrix} 1.1\,c_1 \\ 0.9\,c_2 \end{bmatrix} \arrow[u, "S"']
\end{tikzcd}$$

where $S = \begin{bmatrix} 100 & 200 \\ 300 & 100 \end{bmatrix}$

So $D = S^{-1}AS$

An $n \times n$ matrix A is called diagonizable if A is similar to some diagonal matrix D, that is, if there exists an invertible $n \times n$ S s.t. $S^{-1}AS$ is diagonal.

Find a closed form soln. for $\vec{X}(t)$ in the coyote problem. $\vec{X}(t) = c_1 (1.1)^t \vec{v}_1 + c_2 (0.9)^t \vec{v}_2$

ex2: Diagonalize $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

Thm: The matrix of a lin. trans. WRT an eigenbasis.

Consider a lin. trans. $T(\vec{x}) = A\vec{x}$ where A is a square matrix. Suppose $\mathfrak{D} = \{\vec{v}_1, \dots, \vec{v}_n\}$ is an eigenbasis for T w/ $A\vec{v}_i = \lambda_i \vec{v}_i$. Then the $\mathfrak{D}$-matrix D of T is

$$D = S^{-1}AS = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} \text{ where } S = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_n \\ | & & | \end{bmatrix}$$

matrix D is diagonal, and its diagonal entries are the eigenvalues $\lambda_1, \dots, \lambda_n$ of T.

Thm:

(a) Matrix A is diagonalizable iff $\exists$ an eigenbasis for A.

(b) If an $n \times n$ matrix has n distinct eigenvalues, then A is diagonalizable.

Diagonalization Process of $A_{n \times n}$

(a) Find the eigenvals.

(b) Find each eigenspace.

(c) if the sum of $\dim(E_{\lambda_i}) \neq n$, stop.

(d) else, construct D & S.

Thm: Powers of a Diagonalizable Matrix.

If A can be diagonalized as $A = S D S^{-1}$

Then $A^t = S D^t S^{-1}$.