2.2: Linear Transformations in Geometry

(I) use Mathematica to visualize transformations.

scaling by k & (k, ℓ)

reflections. Goal: visualize transformations

skear Goal: Memorize transformation matrices.

rotations.

(II) General projections of $\vec{x}$ in the direction of the unit vector $\vec{u}$. along L.

From Calc III (12.3)

$$\text{proj}_{\vec{a}}\,\vec{b} = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}|^2}\,\vec{a} = \left(\frac{\vec{a}}{|\vec{a}|}\cdot\vec{b}\right)\frac{\vec{a}}{|\vec{a}|}$$

notice that if $\vec{a}$ is a unit vector, then $\frac{\vec{a}}{|\vec{a}|} = \vec{a}$

Goal: Show $\text{proj}_L(\vec{x})$ is a L.T.

$$\begin{aligned}
\text{proj}_L(\vec{x}) &= \vec{x}^{\parallel} \\
&= (\vec{x}\cdot\vec{u})\,\vec{u} \\
&= \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\cdot\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right)\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\
&= (x_1u_1 + x_2u_2)\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\
&= \begin{bmatrix} x_1u_1^2 + x_2u_1u_2 \\ x_1u_1u_2 + x_2u_2^2 \end{bmatrix} \\
&= x_1\begin{bmatrix} u_1^2 \\ u_1u_2 \end{bmatrix} + x_2\begin{bmatrix} u_1u_2 \\ u_2^2 \end{bmatrix} \\
&= \underbrace{\begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}}_{\text{projection matrix } A}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\end{aligned}$$

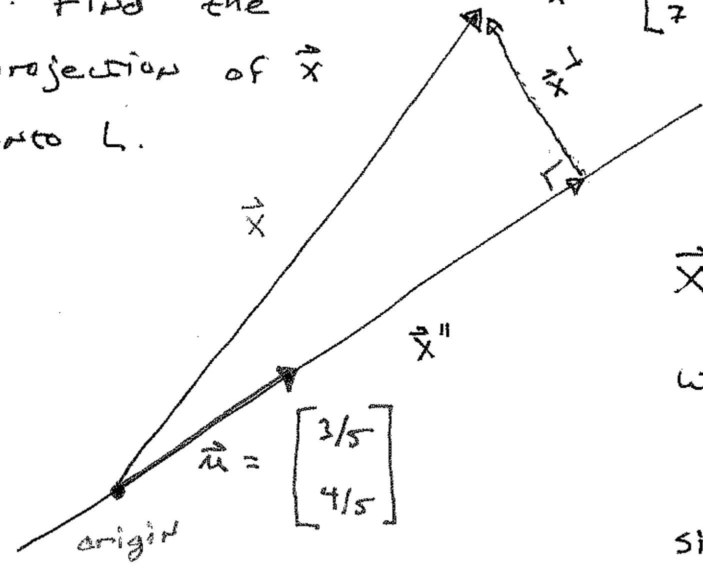
Since $\text{proj}_L(\vec{x}) = A\vec{x}$, we know the projection is a linear transformation.

Goal: use vector concepts to learn about $\vec{x}^{\parallel}$ & $\vec{x}^{\perp}$

ex1: Find the projection of $\vec{x}$ onto L.

$\vec{x} = \begin{bmatrix} 10 \\ 7 \end{bmatrix}$

$L: \; 4x - 3y = 0.$



$\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$

where $\vec{x}^{\parallel} = \text{proj}_L(\vec{x})$

since $\vec{u} \parallel L$, we know that $\vec{x}^{\parallel} = k\vec{u}$ for some scalar k. We must find k.

we know
$$
\begin{aligned}
0 &= \vec{x}^{\perp} \cdot \vec{u} \\
&= (\vec{x} - \vec{x}^{\parallel}) \cdot \vec{u} \\
&= (\vec{x} - k\vec{u}) \cdot \vec{u} \\
&= \vec{x} \cdot \vec{u} - k\|\vec{u}\|^2
\end{aligned}
$$

and so $k = \vec{x} \cdot \vec{u} = \begin{bmatrix} 10 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix} = 58/5$

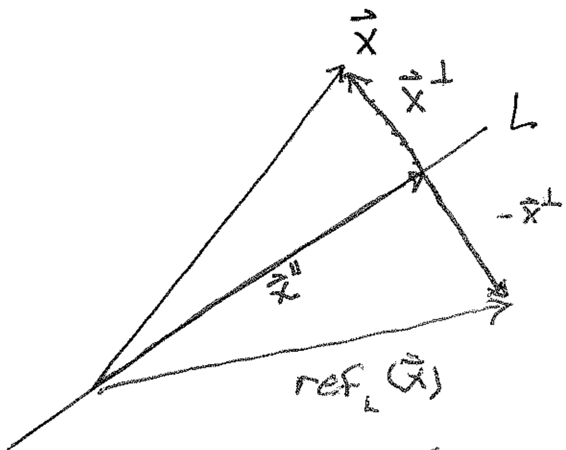
Now $\vec{x}^{\parallel} = k\vec{u} = \frac{58}{5} \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix} = \begin{bmatrix} 174/25 \\ 232/25 \end{bmatrix}$

and $\vec{x}^{\perp} = \vec{x} - \vec{x}^{\parallel} = \begin{bmatrix} 10 \\ 7 \end{bmatrix} - \begin{bmatrix} 174/25 \\ 232/25 \end{bmatrix} = \begin{bmatrix} 76/25 \\ -57/25 \end{bmatrix}$

you can confirm that $\vec{x}^{\parallel} + \vec{x}^{\perp} = \vec{x}$ and $\vec{x}^{\parallel} \cdot \vec{x}^{\perp} = 0$

Goal: Show $ref_L(x)$ is a L.T.

Reflections about the line L.



Two different formulas for the reflection

v1: $ref_L(\vec{x}) = \vec{x}^\parallel - \vec{x}^\perp$

$= (\vec{x} - \vec{x}^\perp) - \vec{x}^\perp$

$= \vec{x} - 2\vec{x}^\perp$ (in terms of $\vec{x}^\perp$)

v2: $ref_L(\vec{x}) = \vec{x}^\parallel - \vec{x}^\perp$

$= \vec{x}^\parallel - (\vec{x} - \vec{x}^\parallel)$

$= 2\vec{x}^\parallel - \vec{x}$

$= 2(\vec{x}\cdot\vec{u})\vec{u} - \vec{x}$. where $\vec{u}$ is a unit vector parallel to L.

show the reflection is a linear trans.

$$= 2\begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= x_1\begin{bmatrix} 2u_1^2 - 1 \\ 2u_1u_2 \end{bmatrix} + x_2\begin{bmatrix} 2u_1u_2 \\ 2u_2^2 - 1 \end{bmatrix}$$

and $u_1^2 + u_2^2 = 1$ since $\vec{u}$ is a unit vector.

$$= x_1\begin{bmatrix} u_1^2 - u_2^2 \\ 2u_1u_2 \end{bmatrix} + x_2\begin{bmatrix} 2u_1u_2 \\ u_2^2 - u_1^2 \end{bmatrix}$$

$$= \begin{bmatrix} a & b \\ b & -a \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$ where $a = u_1^2 - u_2^2$ and $b = 2u_1u_2$

since $ref_L(\vec{x}) = A\vec{x}$ we know the reflection is a linear trans.