8:06

Test 1 – Part A
Dusty Wilson
Math 220

Name: Key

An elegantly executed proof is a poem in all but the form in which it is written.

Morris Kline
1908-1992 (American mathematician)

No work = no credit
No Graphing Calculators

Warm-ups (1 pt each): $I \cdot I^{-1} =$ I $\qquad I \cdot I =$ I $\qquad \begin{bmatrix} 1 & 2 \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} =$ [8]

1.) (1 pts) According to Kline (above), how should a good proof be written? Answer using complete English sentences.

A good proof reads like poetry.

2.) (8 pts) Consider $A = \begin{bmatrix} 0 & -3 & -1 \\ 1 & -3 & -1 \\ 0 & -4 & -1 \end{bmatrix}$

Find A^{-1} if it exists. If it doesn't exist, write the letters of my first name in alphabetical order.

$$\left[\begin{array}{ccc|ccc} 0 & -3 & -1 & 1 & 0 & 0 \\ 1 & -3 & -1 & 0 & 1 & 0 \\ 0 & -4 & -1 & 0 & 0 & 1 \end{array}\right] R_1 \Leftrightarrow R_2$$

$$\Leftrightarrow \left[\begin{array}{ccc|ccc} 1 & -3 & -1 & 0 & 1 & 0 \\ 0 & -3 & -1 & 1 & 0 & 0 \\ 0 & -4 & -1 & 0 & 0 & 1 \end{array}\right] -\tfrac{1}{3}R_2 \to R_2$$

$$\Leftrightarrow \left[\begin{array}{ccc|ccc} 1 & -3 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1/3 & -1/3 & 0 & 0 \\ 0 & -4 & -1 & 0 & 0 & 1 \end{array}\right] R_3 + 4R_2 \to R_3$$

$$\Leftrightarrow \left[\begin{array}{ccc|ccc} 1 & -3 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1/3 & -1/3 & 0 & 0 \\ 0 & 0 & 1/3 & -4/3 & 0 & 1 \end{array}\right] 3R_3 \to R_3$$

$$\Rightarrow \left[\begin{array}{ccc|ccc} 1 & -3 & -1 & 0 & 1 & 0 \\ 0 & 1 & 1/3 & -1/3 & 0 & 0 \\ 0 & 0 & 1 & -4 & 0 & 3 \end{array}\right]$$

$R_2 - \frac{1}{3}R_3 \to R_2$
$R_1 + R_3 \to R_1$

$$\left[\begin{array}{ccc|ccc} 1 & -3 & 0 & -4 & 1 & 3 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -4 & 0 & 3 \end{array}\right]$$

$R_1 + 3R_2 \to R_1$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -4 & 0 & 3 \end{array}\right]$$

$$A^{-1} = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & -1 \\ -4 & 0 & 3 \end{bmatrix}$$

3.) (10 pts) Consider the system of linear equations:

$$\left|\begin{array}{ccccccc} 2x_1 & + & 4x_2 & - & 2x_3 & = & 0 \\ 3x_1 & + & 5x_2 & & & = & 1 \\ 4x_1 & + & 7x_2 & - & x_3 & = & 1 \end{array}\right|$$

a.) Write the associated coefficient matrix A

$$\left[\begin{array}{ccc|c} 2 & 4 & -2 & 0 \\ 3 & 5 & 0 & 1 \\ 4 & 7 & -1 & 1 \end{array}\right]$$

b.) Solve the system using Gauss-Jordan Elimination. Express your solution as a vector. Fractions may be required …

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 3 & 5 & 0 & 1 \\ 4 & 7 & -1 & 1 \end{array}\right] \begin{array}{l} \\ R_2 - 3R_1 \to R_2 \\ R_3 - 4R_1 \to R_3 \end{array}$$

$$\Leftrightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -1 & 3 & 1 \\ 0 & -1 & 3 & 1 \end{array}\right] -R_2 \to R_2$$

$$\Leftrightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & -1 & 3 & 1 \end{array}\right] R_2 + R_3 \to R_3$$

$$\Leftrightarrow \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right] R_1 - 2R_2 \to R_1$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$x_1 = 2 - 5x_3$

$x_2 = -1 + 3x_3$

x_3 : free.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -5 \\ 3 \\ 1 \end{bmatrix}$$

c.) What is the rank of the coefficient matrix A found in (a.)?

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Test 1 – Part B **Name:** ______________________________

Dusty Wilson

Math 220

No work = no credit

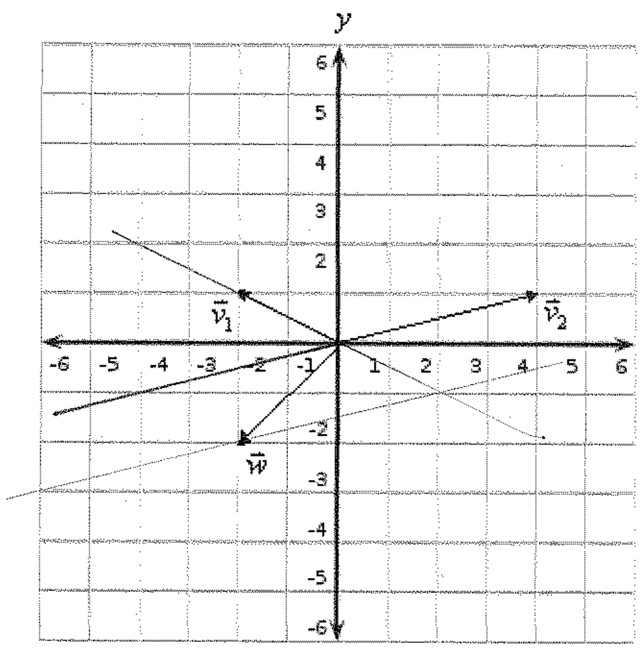
1.) (4 pts) Suppose A is an invertible matrix. Explain two methods for solving $A\vec{x} = \vec{b}$.

(1) Solve directly w/ RREF.

(2) Find A^{-1} & then $\vec{x} = A^{-1}\vec{b}$

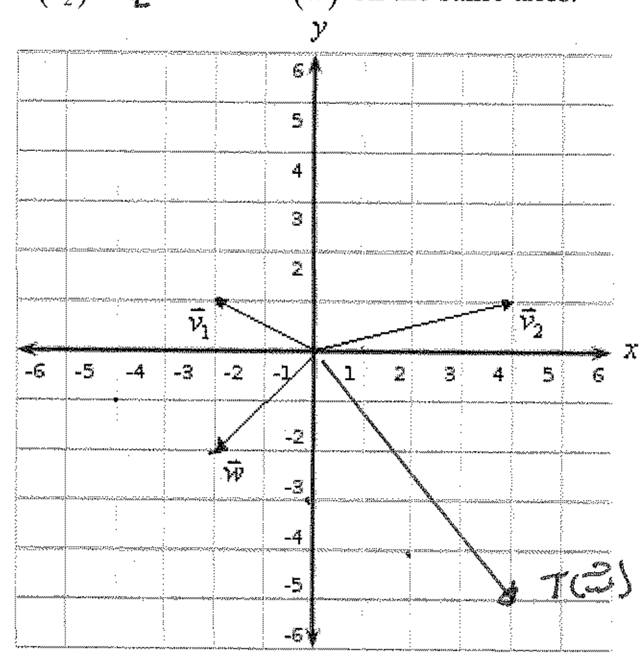
2.) (8 pts) Answer the following:

(a.) Express $\vec{w}$ as a linear combination of $\vec{v}_1$ and $\vec{v}_2$



$\vec{w} = -\vec{v}_1 - \vec{v}_2$

(b.) Consider a linear transformation $T: \mathbb{R}^2 \to \mathbb{R}^2$ such that $T(\vec{v}_1) = 4\vec{v}_2$ and $T(\vec{v}_2) = \vec{v}_2$. Sketch $T(\vec{w})$ on the same axes.



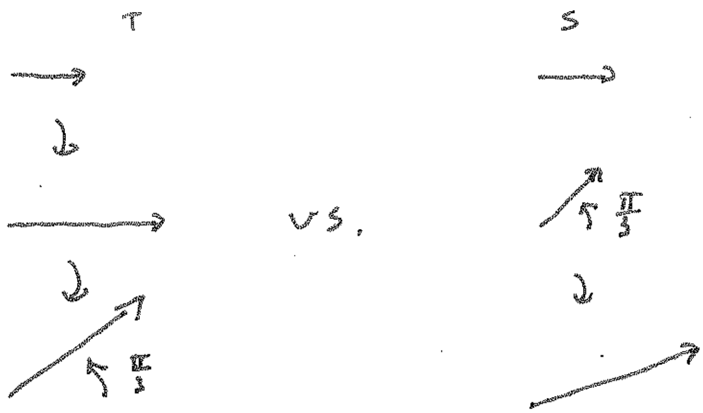
$T(\vec{w}) = T(-\vec{v}_1 - \vec{v}_2)$

$= -T(\vec{v}_1) - T(\vec{v}_2)$

$= -4\vec{v}_2 - \vec{v}_2$

3.) (4 pts) Consider a transformation $T:\mathbb{R}^2 \to \mathbb{R}^2$ that first performs a horizontal stretch by a factor of 2 and then a counter clockwise rotation by $\pi/3$. Is this equivalent to the transformation $S:\mathbb{R}^2 \to \mathbb{R}^2$ that does the rotation first followed by the stretch? Explain your answer.

(Hint: remember the bug).



No. order matters as the pics show.

4.) (4 pts) A linear transformation $T:\mathbb{R}^3 \to \mathbb{R}^5$ can be written as $T(\vec{x}) = A\vec{x}$. What are the dimensions of the matrix A?

$$A_{5\times3}\begin{bmatrix} \\ \\ \end{bmatrix}_{3\times1} = \begin{bmatrix} \\ \\ \\ \\ \end{bmatrix}_{5\times1}$$

5x3

5.) (4 pts) Consider the equation $A_{7\times4}\vec{x} = \vec{0}$.

a.) Is it possible for the equation to have "no solution?" Why or why not?

No ... the homogeneous eqn always has a trivial soln.

b.) What is the greatest number of free variables possible? What is the least?

most: – 4 if A is the zero matrix.

least: 0 if 4 L.I. rows.

6.) (5 pts) Prove that the inverse of a matrix A is unique.

$\square$ proof.

Suppose B and C are inverses of A and $B \neq C$.

$\Rightarrow AB = I = BA$ and $AC = I = CA$ for $B \neq C$

Now $AB = I$

$\Rightarrow C(AB) = CI$

$\Rightarrow (CA)B = C$

$\Rightarrow IB = C$

$\Rightarrow B = C \quad \Rightarrow\Leftarrow$

Hence the inverse is unique.

7.) (8 pts) Regarding linear combinations.

(a.) What is a linear combination? Explain using arbitrary examples (don't assume specific dimensions). A linear combination of vectors is of the form

$c_1 \vec{v}_1 + \dots + c_p \vec{v}_p$

where $c_1, \dots, c_p \in \mathbb{R}$

and $\vec{v}_1, \dots, \vec{v}_p \in \mathbb{R}^m$.

(b.) What is the relationship between linear combinations and matrices?

$$c_1 \vec{v}_1 + \dots + c_p \vec{v}_p = \begin{bmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_p \\ | & | & & | \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix}$$

8.) (8 pts) Suppose $\vec{x} = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$ and L is the line $L: 3y = 4x$. $\vec{u} = \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix}$ is $\parallel$ to $\vec{u}$.

a.) Find the projection of $\vec{x}$ onto L.

$$\text{proj}_L \vec{x} = (\vec{u} \cdot \vec{x}) \vec{u} = \frac{1}{5} \cdot \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix} = \begin{bmatrix} 3/25 \\ 4/25 \end{bmatrix} \leftarrow \vec{x}^{\parallel}.$$

b.) Find the component of $\vec{x}$ perpendicular to the line L.

$$\vec{x}^{\perp} = \vec{x} - \vec{x}^{\parallel} = \begin{bmatrix} 72/25 \\ -54/25 \end{bmatrix}$$