7.6: Stability

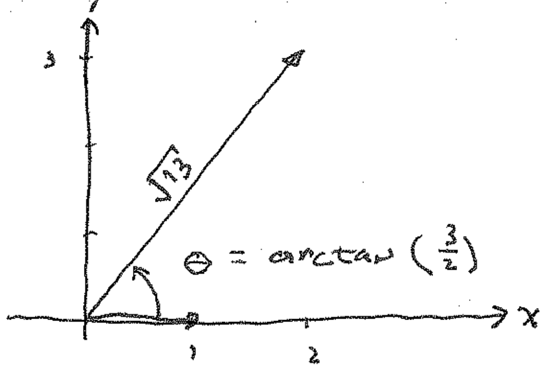
What happens to Dynamical Systems as $t \to \infty$.

<u>ex1</u>: recall $A = \begin{bmatrix} 11 & 6 \\ -15 & -7 \end{bmatrix}$

$$\lambda = 2 \pm 3i$$

$$S^{-1} A S = \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} \quad \text{where} \quad S = \begin{bmatrix} -1 & -3 \\ 0 & 5 \end{bmatrix}$$

(A is similar to a rotation-scaling matrix).



rotate by θ

scale by $\sqrt{13}$

so $A = S \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} S^{-1}$

$$= \sqrt{13}\, S \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} S^{-1}$$

and $A^t = \sqrt{13}^{\,t}\, S \begin{bmatrix} \cos(t\theta) & -\sin(t\theta) \\ \sin(t\theta) & \cos(t\theta) \end{bmatrix} S^{-1}$

If $\vec{X}_0 = \begin{bmatrix} -2 \\ 5 \end{bmatrix}$, find a real closed formula for $\vec{X}(t+1) = A\vec{X}(t)$.

That is, we want $\vec{X}(t) = A^t X_0$

$A^t\vec{X}_0$

t	x	y
0	-2	5
1	8	-5
2	58	-85
3	128	-275
4	-242	5
5	-2632	3595
6	-7382	14315

$$= \sqrt{13}^t S \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \underbrace{S^{-1}\vec{X}_0}_{[X_0]_B}$$ (see note)

$$= \sqrt{13}^t S \begin{bmatrix} c & -s \\ s & c \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$= \sqrt{13}^t \begin{bmatrix} -1 & -3 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} -c-s \\ -s+c \end{bmatrix}$$

$$= \sqrt{13}^t \begin{bmatrix} c+s+3s-3c \\ -5s+5c \end{bmatrix}$$

$$= \sqrt{13}^t \begin{bmatrix} 4s-2c \\ 5c-5s \end{bmatrix}$$

$$= \underbrace{\sqrt{13}^t \begin{bmatrix} 4\sin(t\theta) - 2\cos(t\theta) \\ 5\cos(t\theta) - 5\sin(t\theta) \end{bmatrix}}$$

a real closed formula for $\vec{X}(t)$. Clearly this is not a stable system.

Calculator notes

- $\arctan(3/2) \mapsto \theta$
- parametric mode
- check your graph by discarding $(\sqrt{13}^t)$ to start w/. You should get an ellipse.
- Add $(\sqrt{13})^t$ in and check against the table.

Note: $S^{-1}\vec{X}_0$ is $\vec{X}_0$ in terms of the new coordinate system. From $S = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}$ we have that $\vec{X}_0 = -1\vec{v}_1 + 1\vec{v}_2$ and so $S^{-1}\vec{X}_0 = [\vec{X}_0]_B = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

Thm: Consider a dynamical system $\vec{x}(t+1) = A\vec{x}(t)$.
where A is a real 2×2 matrix w/eigenvalues
$\lambda_{1,2} = p \pm iq$, $q \neq 0$. Let $r = |\lambda_1| = |\lambda_2| = \sqrt{p^2+q^2}$.

$r=1$: $\vec{x}(t)$ lies on an ellipse.

$r>1$ $\vec{x}(t)$ spirals out.

$r<1$ $x(t)$ spirals in to the origin.

ex 2:

We quote from a text on computer graphics (M. Beeler et al., "HAKMEM," MIT Artificial Intelligence Report AIM-239, 1972):

> Here is an elegant way to draw almost circles on a point-plotting display.
>
> ```
> CIRCLE ALGORITHM:
>
> NEW X = OLD X - K*OLD Y;
> NEW Y = OLD Y + K*NEW X.
> ```
>
> This makes a very round ellipse centered at the origin with its size determined by the initial point. The circle algorithm was invented by mistake when I tried to save a register in a display hack!

(In the preceding formula, k is a small number.) Here, a dynamical system is defined in "computer lingo." In our terminology, the formulas are

$$x(t+1) = x(t) - ky(t),$$
$$y(t+1) = y(t) + kx(t+1).$$

a. Find the matrix of this transformation. (Note the entry $x(t+1)$ in the second formula.)

b. Explain why the trajectories are ellipses, as claimed.

(a) $A = \begin{bmatrix} 1 & -k \\ k & 1-k^2 \end{bmatrix}$ check

(b) <u>NTS</u>: A has complex eigenvalues w/ magnitude one.

$$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & -k \\ k & (1-\lambda)-k^2 \end{vmatrix}$$

$$= \lambda^2 + \lambda(k^2-2) + 1$$

$$\Rightarrow \quad \lambda = \frac{(2-k^2) \pm \sqrt{(k^2-2)^2 - 4}}{2}$$

$$= \frac{2-k^2 \pm \sqrt{k^2(k^2-4)}}{2}$$

order changes when i is factored out.

λ is complex if $|k| < 2$.

$$\Rightarrow \quad \lambda = p \pm qi \quad \text{where } p = \frac{2-k^2}{2} \text{ and } q = \frac{k^2(4-k^2)}{2}$$

$$\Rightarrow \quad |\lambda| = \sqrt{p^2+q^2} = 1$$

$\therefore$ $\vec{x}(t)$ lives on an elliptic trajectory when $|k| < 2$.

Key concepts from linear alg.

(1) Find or approx. solutions to linear systems.

(2) Linear transformation

- subspaces
- change of basis to simplify the linear trans.
- dynamical systems.

We are searching in the light.