7.2: The eigenvalue problem.

Dfn: For an $n \times n$ matrix A, find all scalars λ s.t $A\vec{v} = \lambda\vec{v}$ has a non-zero solution $\vec{v}$.

Such a scalar λ is an eigenvalue w/ corresponding eigenvector $\vec{v}$.

Q: What do eigenvectors/values do geometrically?

So how do we ~~solve~~ find them?

solve $A\vec{v} = \lambda\vec{v}$ for $\vec{v} \neq \vec{0}$

$$\Rightarrow \quad Av - \lambda v = \vec{0}$$

$$\Rightarrow \quad (A - \lambda I)\vec{v} = \vec{0}$$

If this is to have nontrivial solutions then $A - \lambda I$ must be singular

Step 1: Find all scalars λ s.t. $A - \lambda I$ is singular ... that is $\det(A - \lambda I) = 0$.

Step 2: Given a scalar λ s.t. $A - \lambda I$ is singular, find all nonzero $\vec{v}$'s s.t. $(A - \lambda I)\vec{v} = \vec{0}$.

ex1: Find the eigenvalues of $A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$

Soln: characteristic poly $\lambda^2 - 4\lambda + 4$

$\lambda = 2$ (alg. mult 2).

If A $(n \times n)$, then $\det(A - \lambda I)$ is a poly w/ degree n. We call it the characteristic poly.

The zeros of the char. poly are the eigenvals.

ex2: Find the eigenvals of $B = \begin{array}{c} \begin{matrix} -\frac{1}{2} & \;\;1 & \;\;0 \end{matrix} \\ \begin{bmatrix} -2 & -1 & 0 \\ 0 & 1 & 1 \\ -2 & -2 & -1 \end{bmatrix} \end{array}$

$\lambda = 0$; $\lambda = -1$ (alg. mult. 2)

By FToA

(a) an $n \times n$ matrix can have no more than n distinct eigen vals.

(b) an $n \times n$ matrix always has @ least one eigenval. (possibly complex)

(c) an $n \times n$ matrix w/ n odd has at least one real eigenval.

<u>Thm</u>: Let A be an $n \times n$ matrix w/ eigenvalue λ

requires induction: give handout

prove it. → (a) λ^k is an eigenval for A^k, $k = 2, 3, \ldots$

→ (b) If A is nonsingular, $\frac{1}{\lambda}$ is an eigenval. of A^{-1}.

(c) If α is a scalar, then $(\lambda + \alpha)$ is an eigenval of $(A + \alpha I)$.

<u>Thm</u>: Let A be an $n \times n$ matrix. Then A and A^T have the same eigenvals.

<u>Thm</u>: Let A be an $n \times n$ matrix. Then A is singular iff $\lambda = 0$ is an eigenval.

<u>Thm</u>: Let T be an $n \times n$ triangular matrix. Then the eigenvals of T are its diag. entries.

Precalculus, Mathematics for Calculus. 6th ed.
Stewart, Redlin, Watson

(a) Draw a time line (as in Example 1) to show that to set up an annuity in perpetuity of amount R per time period, the amount that must be invested now is

$$A_p = \frac{R}{1+i} + \frac{R}{(1+i)^2} + \frac{R}{(1+i)^3} + \cdots + \frac{R}{(1+i)^n} + \cdots$$

where i is the interest rate per time period.

(b) Find the sum of the infinite series in part (a) to show that

$$A_p = \frac{R}{i}$$

(c) How much money must be invested now at 10% per year, compounded annually, to provide an annuity in perpetuity of \$5000 per year? The first payment is due in one year.

(d) How much money must be invested now at 8% per year, compounded quarterly, to provide an annuity in perpetuity of \$3000 per year? The first payment is due in one year.

31. **Amortizing a Mortgage** When they bought their house, John and Mary took out a \$90,000 mortgage at 9% interest, repayable monthly over 30 years. Their payment is \$724.17 per month (check this, using the formula in the text). The bank gave them an **amortization schedule**, which is a table showing how much of each payment is interest, how much goes toward the principal, and the remaining principal after each payment. The table below shows the first few entries in the amortization schedule.

Payment number	Total payment	Interest payment	Principal payment	Remaining principal
1	724.17	675.00	49.17	89,950.83
2	724.17	674.63	49.54	89,901.29
3	724.17	674.26	49.91	89,851.38
4	724.17	673.89	50.28	89,801.10

After 10 years they have made 120 payments and are wondering how much they still owe, but they have lost the amortization schedule.

(a) How much do John and Mary still owe on their mortgage? [*Hint:* The remaining balance is the present value of the 240 remaining payments.]

(b) How much of their next payment is interest, and how much goes toward the principal? [*Hint:* Since $9\% \div 12 = 0.75\%$, they must pay 0.75% of the remaining principal in interest each month.]

12.5 MATHEMATICAL INDUCTION

Conjecture and Proof ▶ Mathematical Induction

There are two aspects to mathematics—discovery and proof—and they are of equal importance. We must discover something before we can attempt to prove it, and we cannot be certain of its truth until it has been proved. In this section we examine the relationship between these two key components of mathematics more closely.

▼ Conjecture and Proof

Let's try a simple experiment. We add more and more of the odd numbers as follows:

$$1 = 1$$
$$1 + 3 = 4$$
$$1 + 3 + 5 = 9$$
$$1 + 3 + 5 + 7 = 16$$
$$1 + 3 + 5 + 7 + 9 = 25$$

What do you notice about the numbers on the right side of these equations? They are, in fact, all perfect squares. These equations say the following:

The sum of the first 1 odd number is 1^2.

The sum of the first 2 odd numbers is 2^2.

The sum of the first 3 odd numbers is 3^2.

The sum of the first 4 odd numbers is 4^2.

The sum of the first 5 odd numbers is 5^2.

Consider the polynomial

$$p(n) = n^2 - n + 41$$

Here are some values of $p(n)$:

$$p(1) = 41 \quad p(2) = 43$$
$$p(3) = 47 \quad p(4) = 53$$
$$p(5) = 61 \quad p(6) = 71$$
$$p(7) = 83 \quad p(8) = 97$$

All the values so far are prime numbers. In fact, if you keep going, you will find that $p(n)$ is prime for all natural numbers up to $n = 40$. It might seem reasonable at this point to conjecture that $p(n)$ is prime for *every* natural number n. But that conjecture would be too hasty, because it is easily seen that $p(41)$ is *not* prime. This illustrates that we cannot be certain of the truth of a statement no matter how many special cases we check. We need a convincing argument—a *proof*—to determine the truth of a statement.

This leads naturally to the following question: Is it true that for every natural number n, the sum of the first n odd numbers is n^2? Could this remarkable property be true? We could try a few more numbers and find that the pattern persists for the first 6, 7, 8, 9, and 10 odd numbers. At this point we feel quite sure that this is always true, so we make a *conjecture*:

The sum of the first n odd numbers is n^2.

Since we know that the nth odd number is $2n - 1$, we can write this statement more precisely as

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

It is important to realize that this is still a conjecture. We cannot conclude by checking a finite number of cases that a property is true for all numbers (there are infinitely many). To see this more clearly, suppose someone tells us that he has added up the first trillion odd numbers and found that they do *not* add up to 1 trillion squared. What would you tell this person? It would be silly to say that you're sure it's true because you have already checked the first five cases. You could, however, take out paper and pencil and start checking it yourself, but this task would probably take the rest of your life. The tragedy would be that after completing this task, you would still not be sure of the truth of the conjecture! Do you see why?

Herein lies the power of mathematical proof. A **proof** is a clear argument that demonstrates the truth of a statement beyond doubt.

Mathematical Induction

Let's consider a special kind of proof called **mathematical induction.** Here is how it works: Suppose we have a statement that says something about all natural numbers n. For example, for any natural number n, let $P(n)$ be the following statement:

$P(n)$: The sum of the first n odd numbers is n^2

Since this statement is about all natural numbers, it contains infinitely many statements; we will call them $P(1)$, $P(2)$, ...

$P(1)$: The sum of the first 1 odd number is 1^2.

$P(2)$: The sum of the first 2 odd numbers is 2^2.

$P(3)$: The sum of the first 3 odd numbers is 3^2.

$\vdots \qquad \vdots$

How can we prove all of these statements at once? Mathematical induction is a clever way of doing just that.

The crux of the idea is this: Suppose we can prove that whenever one of these statements is true, then the one following it in the list is also true. In other words,

For every k, if $P(k)$ is true, then $P(k + 1)$ is true.

This is called the **induction step** because it leads us from the truth of one statement to the truth of the next. Now suppose that we can also prove that

$P(1)$ is true.

The induction step now leads us through the following chain of statements:

$P(1)$ is true, so $P(2)$ is true.

$P(2)$ is true, so $P(3)$ is true.

$P(3)$ is true, so $P(4)$ is true.

$\vdots \qquad \vdots$

So we see that if both the induction step and $P(1)$ are proved, then statement $P(n)$ is proved for all n. Here is a summary of this important method of proof.

PRINCIPLE OF MATHEMATICAL INDUCTION

For each natural number n, let $P(n)$ be a statement depending on n. Suppose that the following two conditions are satisfied.

1. $P(1)$ is true.
2. For every natural number k, if $P(k)$ is true then $P(k + 1)$ is true.

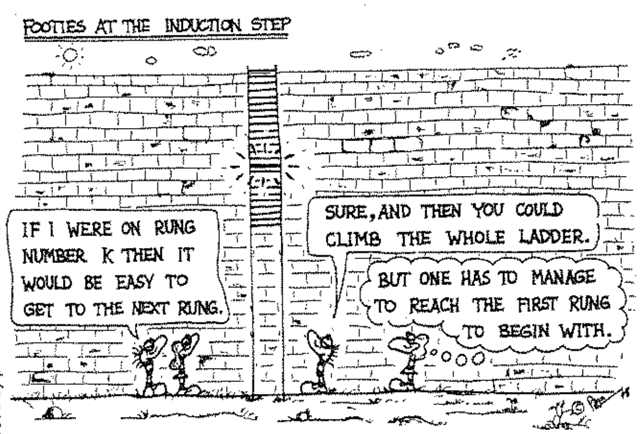
Then $P(n)$ is true for all natural numbers n.

To apply this principle, there are two steps:

Step 1 Prove that $P(1)$ is true.

Step 2 Assume that $P(k)$ is true, and use this assumption to prove that $P(k + 1)$ is true.

Notice that in Step 2 we do not prove that $P(k)$ is true. We only show that *if* $P(k)$ is true, *then* $P(k + 1)$ is also true. The assumption that $P(k)$ is true is called the **induction hypothesis**.



We now use mathematical induction to prove that the conjecture that we made at the beginning of this section is true.

EXAMPLE 1 | A Proof by Mathematical Induction

Prove that for all natural numbers n,

$$1 + 3 + 5 + \cdots + (2n - 1) = n^2$$

SOLUTION Let $P(n)$ denote the statement $1 + 3 + 5 + \cdots + (2n - 1) = n^2$.

Step 1 We need to show that $P(1)$ is true. But $P(1)$ is simply the statement that $1 = 1^2$, which is of course true.

Step 2 We assume that $P(k)$ is true. Thus our induction hypothesis is

$$1 + 3 + 5 + \cdots + (2k - 1) = k^2$$

We want to use this to show that $P(k + 1)$ is true, that is,

$$1 + 3 + 5 + \cdots + (2k - 1) + [2(k + 1) - 1] = (k + 1)^2$$

[Note that we get $P(k + 1)$ by substituting $k + 1$ for each n in the statement $P(n)$.] We start with the left side and use the induction hypothesis to obtain the right side of the equation:

This equals k^2 by the induction hypothesis

$$
\begin{aligned}
&1 + 3 + 5 + \cdots + (2k - 1) + [2(k + 1) - 1] \\
&= [1 + 3 + 5 + \cdots + (2k - 1)] + [2(k + 1) - 1] && \text{Group the first } k \text{ terms} \\
&= k^2 + [2(k + 1) - 1] && \text{Induction hypothesis} \\
&= k^2 + [2k + 2 - 1] && \text{Distributive Property} \\
&= k^2 + 2k + 1 && \text{Simplify} \\
&= (k + 1)^2 && \text{Factor}
\end{aligned}
$$

Thus $P(k + 1)$ follows from $P(k)$, and this completes the induction step.

Having proved Steps 1 and 2, we conclude by the Principle of Mathematical Induction that $P(n)$ is true for all natural numbers n.

NOW TRY EXERCISE **3**

EXAMPLE 2 | A Proof by Mathematical Induction

Prove that for every natural number n,

$$1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2}$$

SOLUTION Let $P(n)$ be the statement $1 + 2 + 3 + \cdots + n = n(n + 1)/2$. We want to show that $P(n)$ is true for all natural numbers n.

Step 1 We need to show that $P(1)$ is true. But $P(1)$ says that

$$1 = \frac{1(1 + 1)}{2}$$

and this statement is clearly true.

Step 2 Assume that $P(k)$ is true. Thus our induction hypothesis is

$$1 + 2 + 3 + \cdots + k = \frac{k(k + 1)}{2}$$

We want to use this to show that $P(k + 1)$ is true, that is,

$$1 + 2 + 3 + \cdots + k + (k + 1) = \frac{(k + 1)[(k + 1) + 1]}{2}$$

So we start with the left side and use the induction hypothesis to obtain the right side:

This equals $\frac{k(k + 1)}{2}$ by the induction hypothesis

$$
\begin{aligned}
&1 + 2 + 3 + \cdots + k + (k + 1) \\
&= [1 + 2 + 3 + \cdots + k] + (k + 1) && \text{Group the first } k \text{ terms} \\
&= \frac{k(k + 1)}{2} + (k + 1) && \text{Induction hypothesis} \\
&= (k + 1)\left(\frac{k}{2} + 1\right) && \text{Factor } k + 1 \\
&= (k + 1)\left(\frac{k + 2}{2}\right) && \text{Common denominator} \\
&= \frac{(k + 1)[(k + 1) + 1]}{2} && \text{Write } k + 2 \text{ as } k + 1 + 1
\end{aligned}
$$

Thus $P(k + 1)$ follows from $P(k)$, and this completes the induction step.

Having proved Steps 1 and 2, we conclude by the Principle of Mathematical Induction that $P(n)$ is true for all natural numbers n.

NOW TRY EXERCISE 5 ■

The following box gives formulas for the sums of powers of the first n natural numbers. These formulas are important in calculus. Formula 1 is proved in Example 2. The other formulas are also proved by using mathematical induction (see Exercises 6 and 9).

SUMS OF POWERS

0. $\sum_{k=1}^{n} 1 = n$

1. $\sum_{k=1}^{n} k = \dfrac{n(n+1)}{2}$

2. $\sum_{k=1}^{n} k^2 = \dfrac{n(n+1)(2n+1)}{6}$

3. $\sum_{k=1}^{n} k^3 = \dfrac{n^2(n+1)^2}{4}$

It might happen that a statement $P(n)$ is false for the first few natural numbers but true from some number on. For example, we might want to prove that $P(n)$ is true for $n \geq 5$. Notice that if we prove that $P(5)$ is true, then this fact, together with the induction step, would imply the truth of $P(5), P(6), P(7), \ldots$. The next example illustrates this point.

EXAMPLE 3 | Proving an Inequality by Mathematical Induction

Prove that $4n < 2^n$ for all $n \geq 5$.

SOLUTION Let $P(n)$ denote the statement $4n < 2^n$.

Step 1 $P(5)$ is the statement that $4 \cdot 5 < 2^5$, or $20 < 32$, which is true.

Step 2 Assume that $P(k)$ is true. Thus our induction hypothesis is

$$4k < 2^k$$

We want to use this to show that $P(k+1)$ is true, that is,

$$4(k+1) < 2^{k+1}$$

We get $P(k+1)$ by replacing n by $k+1$ in the statement $P(n)$.

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BLAISE PASCAL (1623–1662) is considered one of the most versatile minds in modern history. He was a writer and philosopher as well as a gifted mathematician and physicist. Among his contributions that appear in this book are Pascal's triangle and the Principle of Mathematical Induction.

Pascal's father, himself a mathematician, believed that his son should not study mathematics until he was 15 or 16. But at age 12, Blaise insisted on learning geometry and proved most of its elementary theorems himself. At 19 he invented the first mechanical adding machine. In 1647, after writing a major treatise on the conic sections, he abruptly abandoned mathematics because he felt that his intense studies were contributing to his ill health. He devoted himself instead to frivolous recreations such as gambling, but this only served to pique his interest in probability. In 1654 he miraculously survived a carriage accident in which his horses ran off a bridge. Taking this to be a sign from God, Pascal entered a monastery, where he pursued theology and philosophy, writing his famous *Pensées*. He also continued his mathematical research. He valued faith and intuition more than reason as the source of truth, declaring that "the heart has its own reasons, which reason cannot know."

So we start with the left-hand side of the inequality and use the induction hypothesis to show that it is less than the right-hand side. For $k \geq 5$ we have

$$\begin{aligned} 4(k+1) &= 4k + 4 && \text{Distributive Property} \\ &< 2^k + 4 && \text{Induction hypothesis} \\ &< 2^k + 4k && \text{Because } 4 < 4k \\ &< 2^k + 2^k && \text{Induction hypothesis} \\ &= 2 \cdot 2^k && \\ &= 2^{k+1} && \text{Property of exponents} \end{aligned}$$

Thus $P(k+1)$ follows from $P(k)$, and this completes the induction step.

Having proved Steps 1 and 2, we conclude by the Principle of Mathematical Induction that $P(n)$ is true for all natural numbers $n \geq 5$.

NOW TRY EXERCISE **21**

12.5 EXERCISES

CONCEPTS

1. Mathematical induction is a method of proving that a statement $P(n)$ is true for all ________ numbers n. In Step 1 we prove that ________ is true.

2. Which of the following is true about Step 2 in a proof by mathematical induction?
(i) We prove "$P(k+1)$ is true."
(ii) We prove "If $P(k)$ is true, then $P(k+1)$ is true."

SKILLS

3–14 ■ Use mathematical induction to prove that the formula is true for all natural numbers n.

3. $2 + 4 + 6 + \cdots + 2n = n(n+1)$

4. $1 + 4 + 7 + \cdots + (3n-2) = \dfrac{n(3n-1)}{2}$

5. $5 + 8 + 11 + \cdots + (3n+2) = \dfrac{n(3n+7)}{2}$

6. $1^2 + 2^2 + 3^2 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

7. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n(n+1) = \dfrac{n(n+1)(n+2)}{3}$

8. $1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + n(n+2) = \dfrac{n(n+1)(2n+7)}{6}$

9. $1^3 + 2^3 + 3^3 + \cdots + n^3 = \dfrac{n^2(n+1)^2}{4}$

10. $1^3 + 3^3 + 5^3 + \cdots + (2n-1)^3 = n^2(2n^2-1)$

11. $2^3 + 4^3 + 6^3 + \cdots + (2n)^3 = 2n^2(n+1)^2$

12. $\dfrac{1}{1 \cdot 2} + \dfrac{1}{2 \cdot 3} + \dfrac{1}{3 \cdot 4} + \cdots + \dfrac{1}{n(n+1)} = \dfrac{n}{(n+1)}$

13. $1 \cdot 2 + 2 \cdot 2^2 + 3 \cdot 2^3 + 4 \cdot 2^4 + \cdots + n \cdot 2^n$
$= 2[1 + (n-1)2^n]$

14. $1 + 2 + 2^2 + \cdots + 2^{n-1} = 2^n - 1$

15. Show that $n^2 + n$ is divisible by 2 for all natural numbers n.

16. Show that $5^n - 1$ is divisible by 4 for all natural numbers n.

17. Show that $n^2 - n + 41$ is odd for all natural numbers n.

18. Show that $n^3 - n + 3$ is divisible by 3 for all natural numbers n.

19. Show that $8^n - 3^n$ is divisible by 5 for all natural numbers n.

20. Show that $3^{2n} - 1$ is divisible by 8 for all natural numbers n.

21. Prove that $n < 2^n$ for all natural numbers n.

22. Prove that $(n+1)^2 < 2n^2$ for all natural numbers $n \geq 3$.

23. Prove that if $x > -1$, then $(1+x)^n \geq 1 + nx$ for all natural numbers n.

24. Show that $100n \leq n^2$ for all $n \geq 100$.

25. Let $a_{n+1} = 3a_n$ and $a_1 = 5$. Show that $a_n = 5 \cdot 3^{n-1}$ for all natural numbers n.