5.4: Least Squares.

Ex1: Find a linear model to describe height as a fct of armspan.

$$c_0 R + c_1 = H$$

collect student data.

$$\underset{A}{\begin{bmatrix} R_1 & 1 \\ R_2 & 1 \\ R_3 & 1 \\ & \vdots \end{bmatrix}} \underset{\vec{x}}{\begin{bmatrix} c_0 \\ c_1 \end{bmatrix}} = \underset{\vec{b}}{\begin{bmatrix} H_1 \\ H_2 \\ H_3 \\ \vdots \end{bmatrix}}$$

can we solve $A\vec{x} = \vec{b}$? ... no, it's inconsistent.

since we can't find a perfect fit, let's look for the "best possible" $\vec{x}$ ... call it $\vec{x}^*$.

$\vec{x}^* = (A^TA)^{-1}A^T\vec{b}$ or the solution to

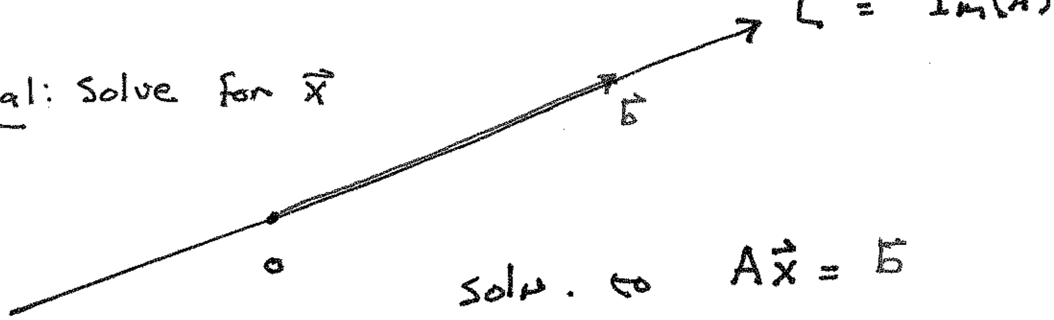
$A^TA\,\vec{x}^* = A^T\vec{b}$ ....

compare to the least squares sol. on the calculator...

wow! where did this come from?

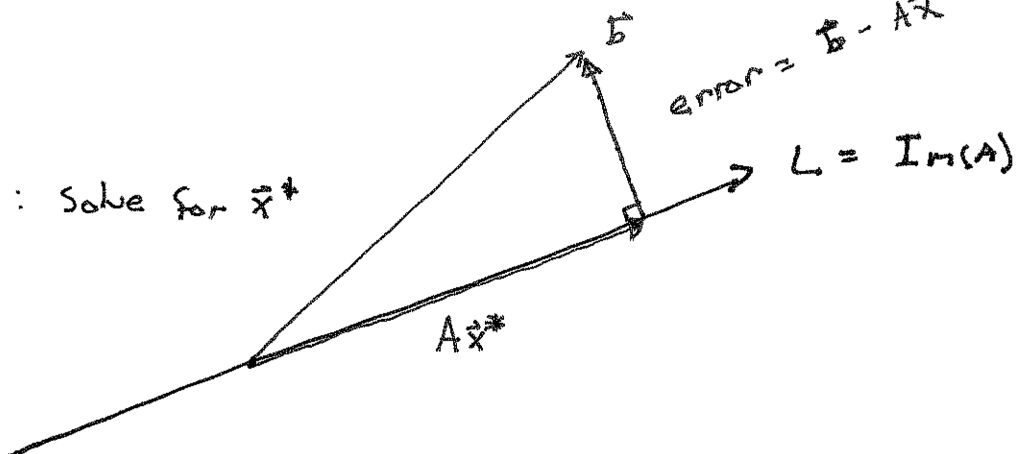
In 2D: $A : \mathbb{R}^2 \to \mathbb{R}^2$

Goal: Solve for $\vec{x}$



soln. to $A\vec{x} = \vec{b}$
is the $\vec{x}$ s.t. $A\vec{x} = \vec{b}$

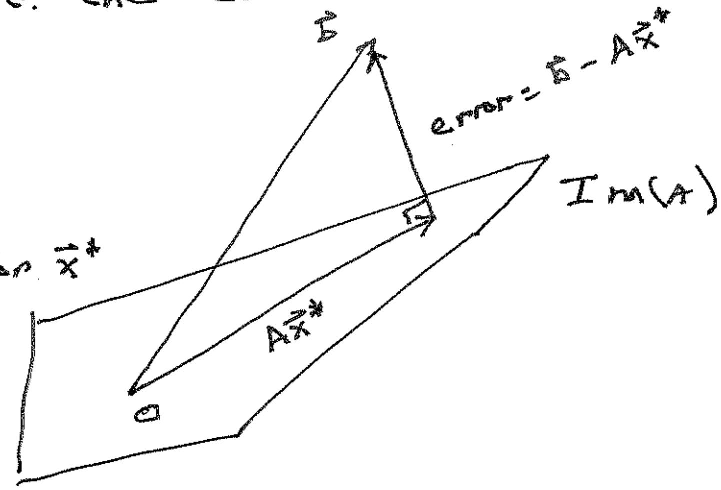
Goal: Solve for $\vec{x}^*$



No soln. to $A\vec{x} = \vec{b}$
The closest vector to $\vec{b}$ in $Im(A)$ is $A\vec{x}^*$ and so we want to find this $\vec{x}^*$ s.t. the error vector $\vec{b} - A\vec{x}^*$ is minimized.

In 3D: $A : \mathbb{R}^3 \mapsto \mathbb{R}^3$

Goal: solve for $\vec{x}^*$



consider $V = \text{Im}(A)$ where $A = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix}$

$$V^{\perp} = \{ \vec{x} \in \mathbb{R}^n \mid \vec{v}_i^{\,T} \vec{x} = 0 \quad \text{for } i = 1, \dots, m \}$$

The solutions to

$$\Rightarrow \begin{bmatrix} - & \vec{v}_1^{\,T} & - \\ & \vdots & \\ - & \vec{v}_m^{\,T} & - \end{bmatrix} \vec{x} = \vec{0} \quad \text{make up } V^{\perp}.$$

$$\Rightarrow \quad V^{\perp} = (\text{Im}(A))^{\perp} = \text{Ker}(A^T).$$

<u>Thm</u>:

(a) if A is an $n \times m$ matrix, $\ker(A) = \ker(A^T A)$.

* key concept in the proof:

If $\ker(A) \subseteq \ker(A^T A)$ and $\ker(A^T A) \subseteq \ker(A)$

Then $\ker(A) = \ker(A^T A)$.

(b) if A is an $n \times m$ matrix w/ $\ker(A) = \{\vec{0}\}$

Then $A^T A$ is invertible.

* This follows from A.

<u>Dfn</u>: Consider a lin. sys. $A\vec{x} = \vec{b}$ where A is an $n \times m$ matrix. A vector $\vec{x}^* \in \mathbb{R}^m$ is called a least-squares sol. of this system if

$$\| \vec{b} - A\vec{x}^* \| \leq \| \vec{b} - A\vec{x} \| \quad \text{for all } \vec{x} \in \mathbb{R}^m$$

Q: why is this called the "least-squares" soln.?

Logic

We want $\vec{x}^*$ sol. to $A\vec{x}=\vec{b}$

$$\Updownarrow$$

$\|\vec{b}-A\vec{x}^*\| \leq \|\vec{b}-A\vec{x}\|$ for all $\vec{x}\in\mathbb{R}^n$

$$\Updownarrow$$

note: $A\vec{x}^* = \text{proj}_V \vec{b}$

$\vec{b}-A\vec{x}^* \in V^\perp = (\text{Im}(A))^\perp = \ker(A^T)$

$$\Updownarrow$$

$A^T(\vec{b}-A\vec{x}^*) = \vec{0}$

$$\Updownarrow$$

$A^TAx^* = A^T\vec{b}$. (the normal eqn to $A\vec{x}=\vec{b}$).

If the cols of $A = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix}$ form a basis for $V \Rightarrow \ker(A)=\{\vec{0}\} \Rightarrow A^TA$ is invertible.

$\Rightarrow \vec{x}^* = (A^TA)^{-1}A^T\vec{b}$

and $\text{proj}_V \vec{b} = Ax^* = A(A^TA)^{-1}A^T\vec{b}$.

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<u>Thm:</u> Consider a subspace V of $\mathbb{R}^n$ w/ basis $\vec{v}_1, \dots, \vec{v}_m$. Let $A = \begin{bmatrix} | & & | \\ \vec{v}_1 & \cdots & \vec{v}_m \\ | & & | \end{bmatrix}$.

Then the matrix of the of the orthogonal projection onto V is $A(A^TA)^{-1}A^T$