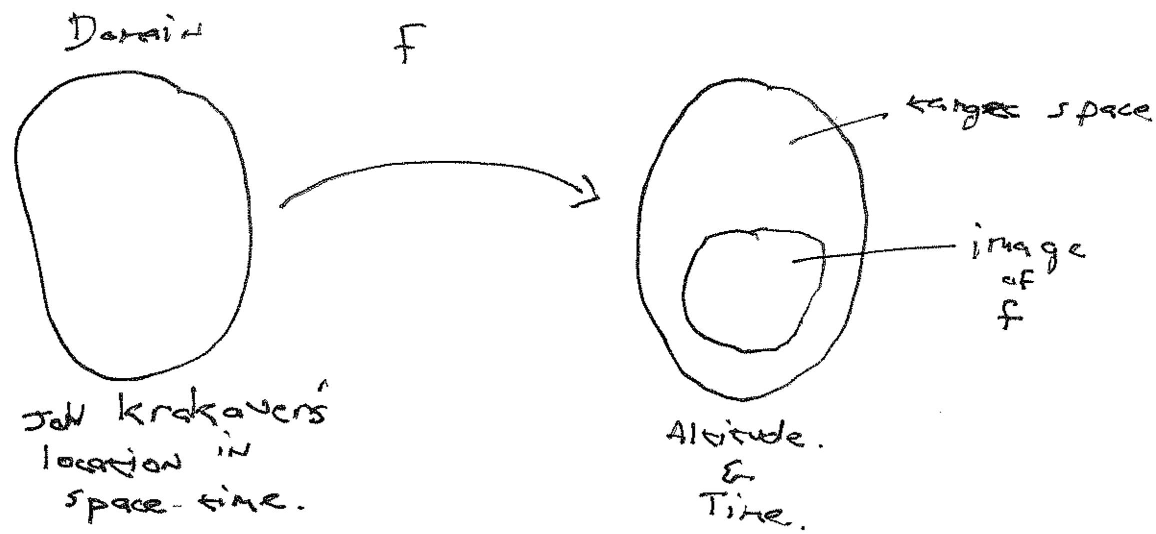
3.1: Subspaces of $\mathbb{R}^n$ & Their Dimension.

Ex 1: Into Thin Air by Jon Krakauer



$f: (\text{long}, \text{lat}, \text{alt.}, \text{time}) \longrightarrow (\text{time}, \text{alt}).$

Note: f is not 1-1.

$\boxed{\text{I}}$ Dfn. If $f: X \mapsto Y$, then

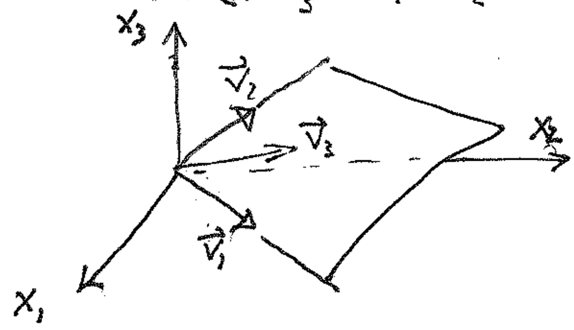
$$\text{image}(f) = \{f(x) \mid x \in X\}$$
$$= \{b \in Y \mid b = f(x) \text{ for some } x \in X\}$$

Ex 2: Find the image of the linear transformation $T(\vec{x}) = A\vec{x}$ where

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \Rightarrow A\vec{x} = \overset{\vec{v}_1}{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} x_1 + \overset{\vec{v}_2}{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}} x_2 + \overset{\vec{v}_3}{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}} x_3$$

NOTE: $\vec{v}_3 = \vec{v}_1 + \vec{v}_2$

The image of T is the plane spanned by $\vec{v}_1, \vec{v}_2, \vec{v}_3$



<u>Dfn</u>. The set of all linear combinations $c_1\vec{v}_1 + \dots + c_m\vec{v}_m$ of $\vec{v}_1, \dots, \vec{v}_m \in \mathbb{R}^n$ is called the <u>span</u>: $\text{span}(\vec{v}_1, \dots, \vec{v}_m)$.

<u>Thm</u>: The image of the linear transformation $T(\vec{x}) = A\vec{x}$ is the span of the columns of A. This is why the image is sometimes called the column space. (see ex2).

- $\vec{0} \in \text{Im}(T)$.
- T is closed under vector addition and scalar multiplication.

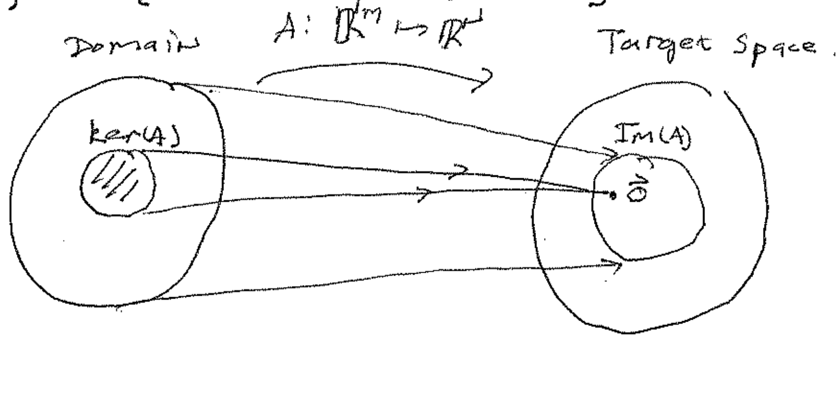
$\boxed{\text{II}}$ The kernel is similar to the set of x intercepts of a fct.

<u>Ex 3</u>: Solve $A\vec{x} = \vec{0}$ where $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \\ 3 & 6 & 4 & 7 \end{bmatrix}$ to find the kernel(A).

$$\text{rref}([A|0]) = \left[\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \text{so } \vec{x} = s\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ -1 \\ +1 \end{bmatrix}$$

<u>check</u>: $A\vec{0}$, $A\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $A\begin{bmatrix} -3 \\ 1 \\ -1 \\ 1 \end{bmatrix}$

<u>Dfn</u>: $\ker(T) = \{\vec{x} \in \mathbb{R}^m \mid T(\vec{x}) = \vec{0}\}$



<u>Thm</u>: Properties of the kernel.

consider a lin. trans. T from $\mathbb{R}^m$ to $\mathbb{R}^n$.

- $\vec{0}_m \in \ker(T)$
- The kernel is closed under addition & scalar mult.

$\square$ proof of closure under addition.

Let $\vec{v}, \vec{w} \in \ker(T)$.

$\Rightarrow T(\vec{v}) = T(\vec{w}) = \vec{0}$.

But $T(\vec{v}+\vec{w}) = T(\vec{v}) + T(\vec{w}) = \vec{0} + \vec{0} = \vec{0}$

so $\vec{v}+\vec{w} \in \ker(T)$ $\blacksquare$

$\boxed{\text{III}}$ Invertible matrices.

- ~~If~~ A is invertible, iff $A\vec{x} = \vec{0}$ has only one solution, $\vec{x} = \vec{0}$, so $\ker(A) = \{\vec{0}\}$
- If $A_{n \times m}$, then $\ker(A) = \{\vec{0}\}$ iff $\operatorname{rank}(A) = m$
- If $A_{n \times m}$ and $m > n$, then there are nonzero vecs. in $\ker(A)$. (see ex. 3).
- $A_{n \times n}$ ... these are equivalent.
  - (i) A is invertible
  - (ii) $A\vec{x} = \vec{b}$ has a unique sol. $\forall \vec{b} \in \mathbb{R}^n$
  - (iii) $\operatorname{rref}(A) = I_n$
  - (iv) $\operatorname{rank}(A) = n$
  - (v) $\operatorname{im}(A) = \mathbb{R}^n$
  - (vi) $\ker(A) = \{\vec{0}\}$.