2.2: Linear Transformations in Geometry

use Mathematica to visualize transformations.

scaling by k & (k, l)

reflections.

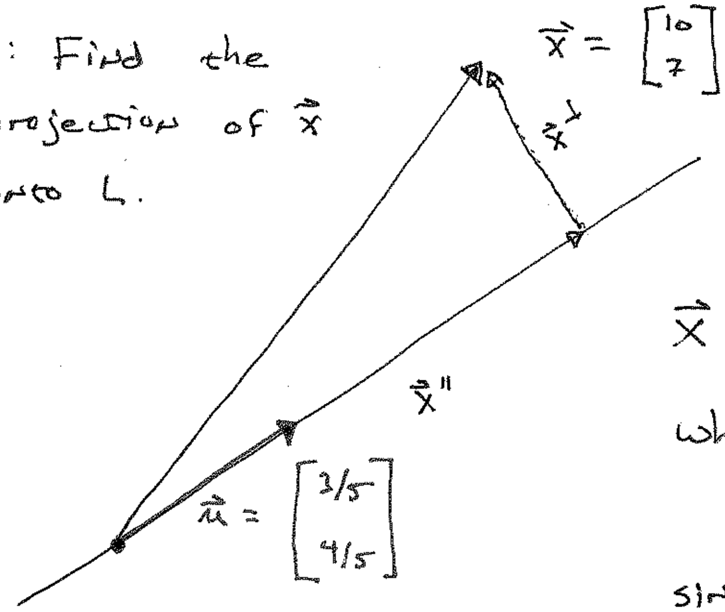
skear

rotations.

General projections of $\vec{x}$ in the direction of the unit vector $\vec{u}$. along L.

$$
\begin{aligned}
\text{proj}_L(\vec{x}) &= \vec{x}^{\parallel} \\
&= (\vec{x} \cdot \vec{u})\, \vec{u} \\
&= \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \right) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\
&= (x_1 u_1 + x_2 u_2) \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\
&= \begin{bmatrix} x_1 u_1^2 + x_2 u_1 u_2 \\ x_1 u_1 u_2 + x_2 u_2^2 \end{bmatrix} \\
&= x_1 \begin{bmatrix} u_1^2 \\ u_1 u_2 \end{bmatrix} + x_2 \begin{bmatrix} u_1 u_2 \\ u_2^2 \end{bmatrix} \\
&= \underbrace{\begin{bmatrix} u_1^2 & u_1 u_2 \\ u_1 u_2 & u_2^2 \end{bmatrix}}_{\text{projection matrix } A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}
\end{aligned}
$$

Since $\text{proj}_L(\vec{x}) = A\vec{x}$, we know the projection is a linear transformati ex1: Find the projection of $\vec{x}$ onto L.



L: $\cancel{3x + 4y}$ $4x - 3y = 0.$

$$\vec{X} = \vec{X}^{\parallel} + \vec{X}^{\perp}$$

where $\vec{X}^{\parallel} = \text{proj}_L(\vec{x})$

since $\vec{u} \parallel L$, we know that $\vec{X}^{\parallel} = k\vec{u}$ for some scalar k. We must find k.

we know
$$0 = \vec{X}^{\perp} \cdot \vec{u}$$
$$= (\vec{x} - \vec{x}^{\parallel}) \cdot \vec{u}$$
$$= (\vec{X} - k\vec{u}) \cdot \vec{u}$$
$$= \vec{X} \cdot \vec{u} - k\|\vec{u}\|^2$$

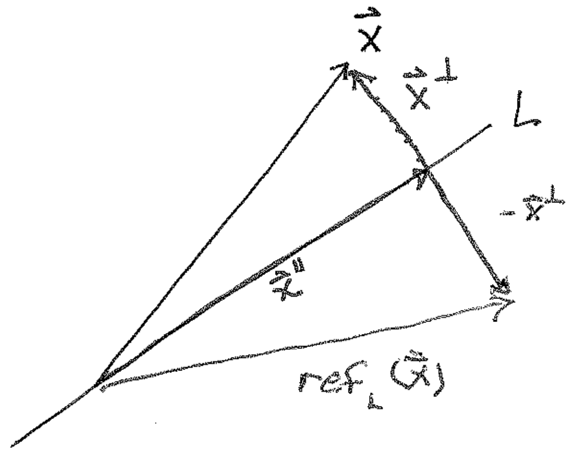
and so $k = \vec{X} \cdot \vec{u} = \begin{bmatrix} 10 \\ 7 \end{bmatrix} \cdot \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix} = 58/5$

Now $\vec{X}^{\parallel} = k\vec{u} = \frac{58}{5} \begin{bmatrix} 3/5 \\ 4/5 \end{bmatrix} = \begin{bmatrix} 174/25 \\ 232/25 \end{bmatrix}$

and $\vec{X}^{\perp} = \vec{X} - \vec{X}^{\parallel} = \begin{bmatrix} 10 \\ 7 \end{bmatrix} - \begin{bmatrix} 174/25 \\ 232/25 \end{bmatrix} = \begin{bmatrix} 76/25 \\ -57/25 \end{bmatrix}$

you can confirm that $\vec{X}^{\parallel} + \vec{X}^{\perp} = \vec{X}$ and $\vec{X}^{\parallel} \cdot \vec{X}^{\perp} = 0$

Reflections about the line L.



v1: $$\begin{aligned} ref_L(\vec{x}) &= \vec{x}^{\parallel} - \vec{x}^{\perp} \\ &= (\vec{x} - \vec{x}^{\perp}) - \vec{x}^{\perp} \\ &= \vec{x} - 2\vec{x}^{\perp} \quad \text{(in terms of } \vec{x}^{\perp}) \end{aligned}$$

v2: $$\begin{aligned} ref_L(\vec{x}) &= \vec{x}^{\parallel} - \vec{x}^{\perp} \\ &= \vec{x}^{\parallel} - (\vec{x} - \vec{x}^{\parallel}) \\ &= 2\vec{x}^{\parallel} - \vec{x} \\ &= 2(\vec{x}\cdot\vec{u})\vec{u} - \vec{x}. \end{aligned}$$ where $\vec{u}$ is a unit vector parallel to L.

show the reflection is a linear trans.

$$= 2\begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$= x_1\begin{bmatrix} 2u_1^2-1 \\ 2u_1u_2 \end{bmatrix} + x_2\begin{bmatrix} 2u_1u_2 \\ 2u_2^2-1 \end{bmatrix}$$

and $u_1^2 + u_2^2 = 1$ since $\vec{u}$ is a unit vector.

$$= x_1\begin{bmatrix} u_1^2-u_2^2 \\ 2u_1u_2 \end{bmatrix} + x_2\begin{bmatrix} 2u_1u_2 \\ u_2^2-u_1^2 \end{bmatrix}$$

$$= \begin{bmatrix} a & b \\ b & -a \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$ where $a = u_1^2 - u_2^2$ and $b = 2u_1u_2$

since $ref_L(\vec{x}) = A\vec{x}$ we know the reflection is a linear trans.