1.3: On the Solutions of Linear Systems; matrix algebra.

I Prelim.

RREF: After Gauss - Jordan Elimination, we say a matrix is in reduced row echelon form, or RREF.

So given matrix A, we can find rref(A).

Rank: We define the rank of matrix A, or rank(A), as the number of leading ones in rref(A).

read carefully thru ex 1-5 and thms 1.3.3 & 1.3.4 on your own.

II Matrix algebra.

- adding matrices (by element)

$$\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & & a_{nm} \end{bmatrix}_{n\times m} + \begin{bmatrix} b_{11} & \cdots & b_{1m} \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nm} \end{bmatrix}_{n\times m} = \begin{bmatrix} a_{11}+b_{11} & \cdots & a_{1m}+b_{1m} \\ \vdots & & \vdots \\ a_{n1}+b_{n1} & \cdots & a_{nm}+b_{nm} \end{bmatrix}_{n\times}$$

multiplying a matrix by a scalar. (by element/entry)

$$k\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \\ a_{n1} & \cdots & a_{nm} \end{bmatrix} = \begin{bmatrix} ka_{11} & \cdots & ka_{1m} \\ \vdots & & \vdots \\ ka_{n1} & \cdots & ka_{nm} \end{bmatrix}$$

matrix multiplication (row by column).

<u>ex1</u>: Find

$$\underset{3\times2}{A}\cdot\underset{2\times4}{B} = \begin{array}{c}\vec{\omega}_1\\ \vec{\omega}_2\\ \vec{\omega}_3\end{array}\begin{bmatrix}2 & 6\\ 0 & 4\\ 1 & 2\end{bmatrix}_{3\times2}\begin{array}{c}\begin{array}{cccc}\vec{v}_1 & \vec{v}_2 & \vec{v}_3 & \vec{v}_4\end{array}\\ \begin{bmatrix}1 & 2 & 0 & 1\\ 4 & 5 & 2 & 3\end{bmatrix}_{2\times4}\end{array}$$

same

dim of product.

we are using $\vec{\omega}_i$'s for rows and $\vec{v}_j$'s for cols.

Notice our old friend the dot product ...

<u>ex2</u>: $\vec{\omega}_1 \cdot \vec{v}_1 = \begin{bmatrix}2 & 6\end{bmatrix} \cdot \begin{bmatrix}1\\4\end{bmatrix} = 26$

(the dot product of a row & col vector is a scalar).

more generally.

$$\underset{n\times m}{A}\cdot\underset{m\times p}{B} = \begin{bmatrix}\vec{\omega}_1\\ \vdots\\ \vec{\omega}_n\end{bmatrix}\cdot\begin{bmatrix}\vec{v}_1 & \cdots & \vec{v}_p\end{bmatrix}$$

$$= \begin{bmatrix}\vec{\omega}_1\cdot\vec{v}_1 & \cdots & \vec{\omega}_1\cdot\vec{v}_p\\ \vdots & & \vdots\\ \vec{\omega}_n\cdot\vec{v}_1 & \cdots & \vec{\omega}_n\cdot\vec{v}_p\end{bmatrix}_{n\times p}$$

and if

like $\vec{v}_1$

$$\vec{x} = \begin{bmatrix}x_1\\ \vdots\\ x_m\end{bmatrix}, \text{ then } A\vec{x} = \begin{bmatrix}\vec{\omega}_1\cdot\vec{x}\\ \vdots\\ \vec{\omega}_n\cdot\vec{x}\end{bmatrix}_{n\times1}$$

we want another expression for $A\vec{x}$

<u>ex 3:</u> $A\vec{x} = \begin{bmatrix} 2 & 6 \\ 0 & 4 \\ 1 & 2 \end{bmatrix}_{3\times 2} \begin{bmatrix} 1 \\ 4 \end{bmatrix}_{2\times 1}$

$$= \begin{bmatrix} 2(1) + 6(4) \\ 0(1) + 4(4) \\ 1(1) + 2(4) \end{bmatrix}$$

$$= 1\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + 4\begin{bmatrix} 6 \\ 4 \\ 2 \end{bmatrix}$$

and $A\vec{x} = \begin{bmatrix} \vec{v}_1 & \cdots & \vec{v}_m \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix} = x_1\vec{v}_1 + \dots + x_m\vec{v}_m$

<u>Dfn:</u> A vector $\vec{b} \in \mathbb{R}^n$ is called a linear combination of $\vec{v}_1, \dots, \vec{v}_m \in \mathbb{R}^n$ if there exist scalars $x_1, \dots, x_m$ s.t. $\vec{b} = x_1\vec{v}_1 + \dots + x_m\vec{v}_m$.

so $A\vec{x}$ is a linear combo of $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m$

This will be used extensively in the theoretical areas. we can see ...

$A\vec{x} = x_1\vec{v}_1 + \dots + x_m\vec{v}_m$ (you can break apart $A\vec{x}$)

and

$x_1\vec{v}_1 + \dots + x_n\vec{v}_m = A\vec{x}$ (you can combine a lin. combo into $A\vec{x}$).

Two rules for $A\vec{x}$: If A is an $n\times m$ matrix and $\vec{x},\vec{y}\in\mathbb{R}^n$, and k is a scalar

(a) $A(\vec{x}+\vec{y}) = A\vec{x} + A\vec{y}$

(b) $A(k\vec{x}) = kA\vec{x}$

$\square$ proof of (b).

$$A(k\vec{x}) = \begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix}\begin{bmatrix} kx_1 \\ \vdots \\ kx_m \end{bmatrix} = \begin{bmatrix} a_{11}kx_1 + \dots + a_{1m}kx_m \\ \vdots \\ a_{n1}kx_1 + \dots + a_{nm}kx_m \end{bmatrix}$$

$$= k\begin{bmatrix} a_{11}x_1 + \dots + a_{1m}x_m \\ \vdots \\ a_{n1}x_1 + \dots + a_{nm}x_m \end{bmatrix}$$

$$= k\begin{bmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{bmatrix}\begin{bmatrix} x_1 \\ \vdots \\ x_m \end{bmatrix}$$

$$= kA\vec{x} \quad \blacksquare$$

Linear System: $\left| \begin{array}{l} x_1 + 2x_2 = 8 \\ 3x_1 - x_2 = 3 \end{array} \right|$

w/ augmented matrix $\left[\begin{array}{cc|c} 1 & 2 & 8 \\ 3 & -1 & 3 \end{array}\right]$

$[A \mid b]$

$$\begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \end{bmatrix} \Rightarrow x_1 \begin{bmatrix} 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 8 \\ 3 \end{bmatrix}$$