2.2 Linear Transformations in Geometry

In Example 2.1.5 we saw that the matrix $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ represents a counterclockwise rotation through 90° in the coordinate plane. Many other 2×2 matrices define simple geometrical transformations as well; this section is dedicated to a discussion of some of those transformations.

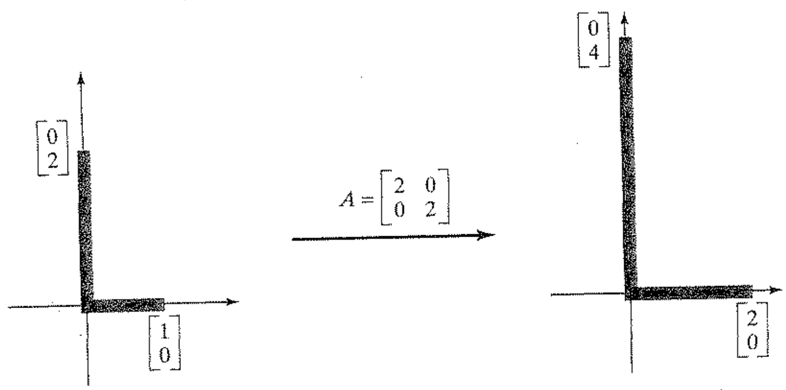
EXAMPLE 1 Consider the matrices

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$D = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad E = \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad F = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}.$$

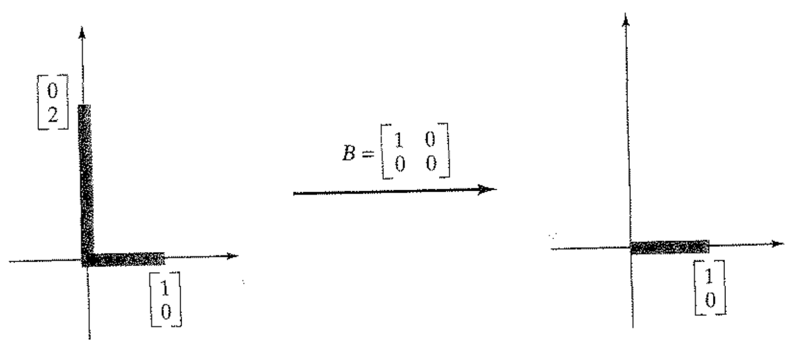
Show the effect of each of these matrices on our standard letter L,[3] and describe each transformation in words.

a.



The L gets enlarged by a factor of 2; we will call this transformation a *scaling* by 2.

b.

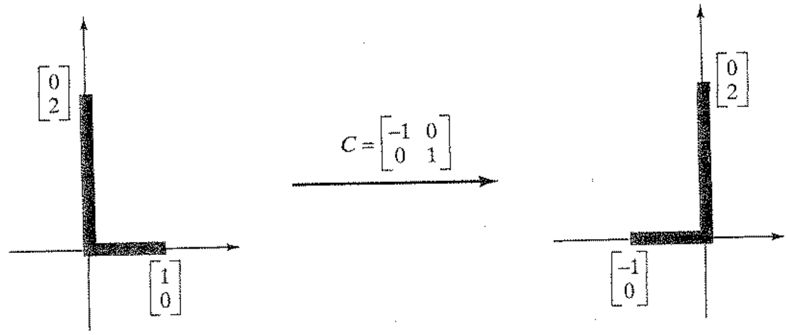


The L gets smashed into the horizontal axis. We will call this transformation the *orthogonal projection onto the horizontal axis*.

---

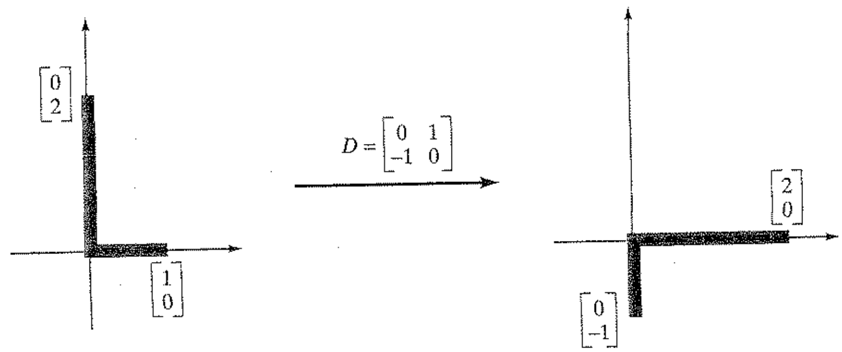
[3] See Example 2.1.5. Recall that vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is the foot of our standard L, and $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ is its back.

c.



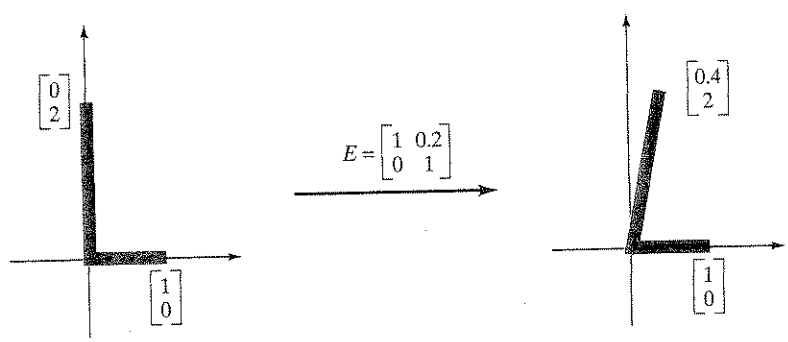
The L gets flipped over the vertical axis. We will call this the *reflection about the vertical axis.*

d.



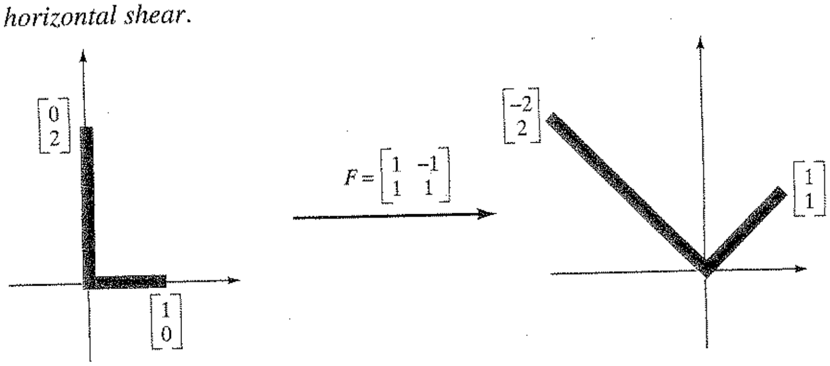
The L is *rotated* through 90°, in the clockwise direction (this amounts to a rotation through −90°). The result is the opposite of what we got in Example 2.1.5.

e.



The foot of the L remains unchanged, while the back is shifted horizontally to the right; the L is italicized, becoming *L*. We will call this transformation a *horizontal shear*.

f.



There are two things going on here: The L is rotated through 45° and also enlarged (scaled) by a factor of $\sqrt{2}$. This is a *rotation combined with a scaling* (you may perform the two transformations in either order). Among all the possible composites of the transformations considered in parts (a) through (e), this one is particularly important in applications as well as in pure mathematics (see Theorem 7.5.3, for example). ■

We will now take a closer look at the six types of transformations we encountered in Example 1.

Scalings

For any positive constant k, the matrix $\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ defines a scaling by k, since

$$\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}\vec{x} = \begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} kx_1 \\ kx_2 \end{bmatrix} = k\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = k\vec{x}.$$

This is a *dilation* (or enlargement) if k exceeds 1, and it is a *contraction* (or shrinking) for values of k between 0 and 1. (What happens when k is negative or zero?)

Orthogonal Projections[4]

Consider a line L in the plane, running through the origin. Any vector $\vec{x}$ in $\mathbb{R}^2$ can be written uniquely as

$$\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp},$$

where $\vec{x}^{\parallel}$ is parallel to line L, and $\vec{x}^{\perp}$ is perpendicular to L. See Figure 1.

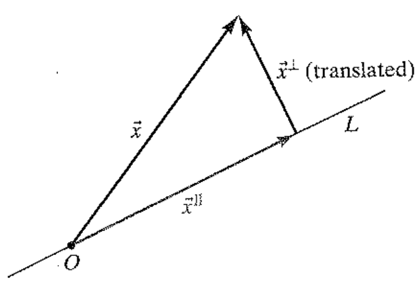


Figure 1

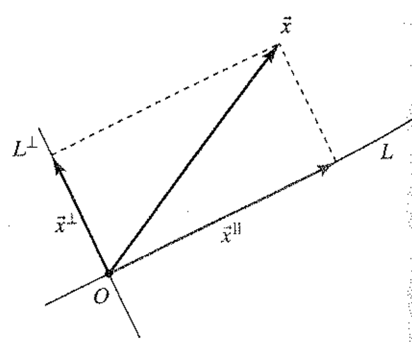


Figure 2

The transformation $T(\vec{x}) = \vec{x}^{\parallel}$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ is called the *orthogonal projection of $\vec{x}$ onto L*, often denoted by $\text{proj}_L(\vec{x})$:

$$\text{proj}_L(\vec{x}) = \vec{x}^{\parallel}.$$

You can think of $\text{proj}_L(\vec{x})$ as the shadow vector $\vec{x}$ casts on L if you shine a light straight down on L.

Let $L^{\perp}$ be the line through the origin perpendicular to L. Note that $\vec{x}^{\perp}$ is parallel to $L^{\perp}$, and we can interpret $\vec{x}^{\perp}$ as the orthogonal projection of $\vec{x}$ onto $L^{\perp}$, as illustrated in Figure 2.

[4]The term *orthogonal* is synonymous with perpendicular. For a more general discussion of projections, see Exercise 33.

We can use the dot product to write a formula for an orthogonal projection. Before proceeding, you may want to review the section "Dot Product, Length, Orthogonality" in the appendix.

To find a formula for $\vec{x}^{\parallel}$, let $\vec{w}$ be a nonzero vector parallel to L. Since $\vec{x}^{\parallel}$ is parallel to $\vec{w}$, we can write

$$\vec{x}^{\parallel} = k\vec{w},$$

for some scalar k about to be determined. Now $\vec{x}^{\perp} = \vec{x} - \vec{x}^{\parallel} = \vec{x} - k\vec{w}$ is perpendicular to line L, that is, perpendicular to $\vec{w}$, meaning that

$$(\vec{x} - k\vec{w}) \cdot \vec{w} = 0.$$

It follows that

$$\vec{x} \cdot \vec{w} - k(\vec{w} \cdot \vec{w}) = 0, \quad \text{or,} \quad k = \frac{\vec{x} \cdot \vec{w}}{\vec{w} \cdot \vec{w}}.$$

We can conclude that

$$\operatorname{proj}_L(\vec{x}) = \vec{x}^{\parallel} = k\vec{w} = \left(\frac{\vec{x} \cdot \vec{w}}{\vec{w} \cdot \vec{w}}\right)\vec{w}.$$

See Figure 3. Consider the special case of a *unit* vector $\vec{u}$ parallel to L. Then the formula for projection simplifies to

$$\operatorname{proj}_L(\vec{x}) = \left(\frac{\vec{x} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\right)\vec{u} = (\vec{x} \cdot \vec{u})\vec{u}$$

since $\vec{u} \cdot \vec{u} = \|\vec{u}\|^2 = 1$ for a unit vector $\vec{u}$.

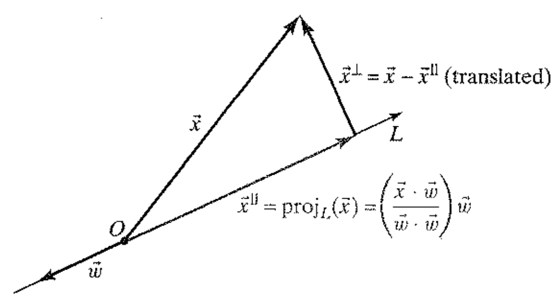


Figure 3

Is the transformation $T(\vec{x}) = \operatorname{proj}_L(\vec{x})$ linear? If so, what is its matrix? If we write

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \text{and} \quad \vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix},$$

then

$$\begin{aligned} \operatorname{proj}_L(\vec{x}) = (\vec{x} \cdot \vec{u})\vec{u} &= \left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \cdot \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}\right)\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ &= (x_1u_1 + x_2u_2)\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ &= \begin{bmatrix} u_1^2x_1 + u_1u_2x_2 \\ u_1u_2x_1 + u_2^2x_2 \end{bmatrix} \\ &= \begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ &= \begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}\vec{x}. \end{aligned}$$

It turns out that $T(\vec{x}) = \text{proj}_L(\vec{x})$ is indeed a linear transformation, with matrix $\begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}$. More generally, if $\vec{w}$ is a nonzero vector parallel to L, then the matrix is $\dfrac{1}{w_1^2+w_2^2}\begin{bmatrix} w_1^2 & w_1w_2 \\ w_1w_2 & w_2^2 \end{bmatrix}$. (See Exercise 12.)

EXAMPLE 2 Find the matrix A of the orthogonal projection onto the line L spanned by $\vec{w} = \begin{bmatrix} 4 \\ 3 \end{bmatrix}$.

Solution

$$A = \frac{1}{w_1^2+w_2^2}\begin{bmatrix} w_1^2 & w_1w_2 \\ w_1w_2 & w_2^2 \end{bmatrix} = \frac{1}{25}\begin{bmatrix} 16 & 12 \\ 12 & 9 \end{bmatrix} = \begin{bmatrix} 0.64 & 0.48 \\ 0.48 & 0.36 \end{bmatrix}$$ ■

Let us summarize our findings.

Definition 2.2.1 **Orthogonal Projections**

Consider a line L in the coordinate plane, running through the origin. Any vector $\vec{x}$ in $\mathbb{R}^2$ can be written uniquely as

$$\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp},$$

where $\vec{x}^{\parallel}$ is parallel to line L, and $\vec{x}^{\perp}$ is perpendicular to L.

The transformation $T(\vec{x}) = \vec{x}^{\parallel}$ from $\mathbb{R}^2$ to $\mathbb{R}^2$ is called the *orthogonal projection of $\vec{x}$ onto L*, often denoted by $\text{proj}_L(\vec{x})$. If $\vec{w}$ is a nonzero vector parallel to L, then

$$\text{proj}_L(\vec{x}) = \left(\frac{\vec{x}\cdot\vec{w}}{\vec{w}\cdot\vec{w}}\right)\vec{w}.$$

In particular, if $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$ is a *unit* vector parallel to L, then

$$\text{proj}_L(\vec{x}) = (\vec{x}\cdot\vec{u})\vec{u}.$$

The transformation $T(\vec{x}) = \text{proj}_L(\vec{x})$ is linear, with matrix

$$\frac{1}{w_1^2+w_2^2}\begin{bmatrix} w_1^2 & w_1w_2 \\ w_1w_2 & w_2^2 \end{bmatrix} = \begin{bmatrix} u_1^2 & u_1u_2 \\ u_1u_2 & u_2^2 \end{bmatrix}.$$

Reflections

Again, consider a line L in the coordinate plane, running through the origin, and let $\vec{x}$ be a vector in $\mathbb{R}^2$. The reflection $\text{ref}_L(\vec{x})$ of $\vec{x}$ about L is shown in Figure 4: We are flipping vector $\vec{x}$ over the line L. The line segment joining the tips of vectors $\vec{x}$ and $\text{ref}_L\vec{x}$ is perpendicular to line L and bisected by L. In previous math courses you have surely seen examples of reflections about the horizontal and vertical axes [when comparing the graphs of $y = f(x)$, $y = -f(x)$, and $y = f(-x)$, for example].

We can use the representation $\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$ to write a formula for $\text{ref}_L(\vec{x})$. See Figure 5.

We can see that

$$\text{ref}_L(\vec{x}) = \vec{x}^{\parallel} - \vec{x}^{\perp}.$$

Alternatively, we can express $\text{ref}_L(\vec{x})$ in terms of $\vec{x}^{\perp}$ alone or in terms of $\vec{x}^{\parallel}$ alone:

$$\text{ref}_L(\vec{x}) = \vec{x}^{\parallel} - \vec{x}^{\perp} = (\vec{x} - \vec{x}^{\perp}) - \vec{x}^{\perp} = \vec{x} - 2\vec{x}^{\perp}$$

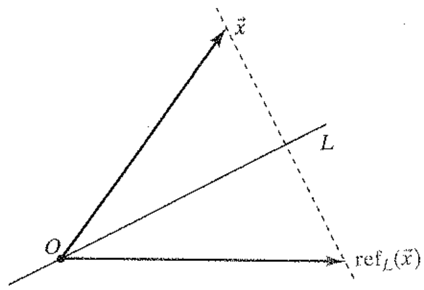


Figure 4

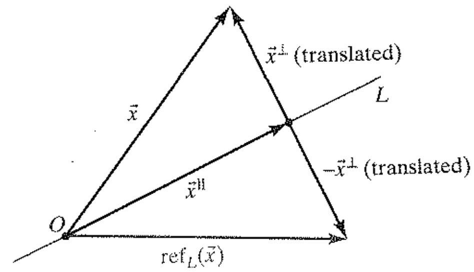


Figure 5

(use Figure 5 to explain this formula geometrically) and

$$\begin{aligned}\text{ref}_L(\vec{x}) &= \vec{x}^{\parallel} - \vec{x}^{\perp} = \vec{x}^{\parallel} - (\vec{x} - \vec{x}^{\parallel}) = 2\vec{x}^{\parallel} - \vec{x} \\ &= 2\text{proj}_L(\vec{x}) - \vec{x} = 2(\vec{x} \cdot \vec{u})\vec{u} - \vec{x},\end{aligned}$$

where $\vec{u}$ is a unit vector parallel to L.

The formula $\text{ref}(\vec{x}) = 2(\vec{x} \cdot \vec{u})\vec{u} - \vec{x}$ allows us to find the matrix of a reflection. It turns out that this matrix is of the form $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$, where $a^2 + b^2 = 1$ (see Exercise 13), and that, conversely, any 2×2 matrix of this form represents a reflection about a line (see Exercise 17).

Definition 2.2.2 Reflections

Consider a line L in the coordinate plane, running through the origin, and let $\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$ be a vector in $\mathbb{R}^2$. The linear transformation $T(\vec{x}) = \vec{x}^{\parallel} - \vec{x}^{\perp}$ is called the *reflection of* $\vec{x}$ *about* L, often denoted by $\text{ref}_L(\vec{x})$:

$$\text{ref}_L(\vec{x}) = \vec{x}^{\parallel} - \vec{x}^{\perp}.$$

We have a formula relating $\text{ref}_L(\vec{x})$ to $\text{proj}_L(\vec{x})$:

$$\text{ref}_L(\vec{x}) = 2\text{proj}_L(\vec{x}) - \vec{x} = 2(\vec{x} \cdot \vec{u})\vec{u} - \vec{x}.$$

The matrix of T is of the form $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$, where $a^2 + b^2 = 1$. Conversely, any matrix of this form represents a reflection about a line.

Use Figure 6 to explain the formula $\text{ref}_L(\vec{x}) = 2\text{proj}_L(\vec{x}) - \vec{x}$ geometrically.

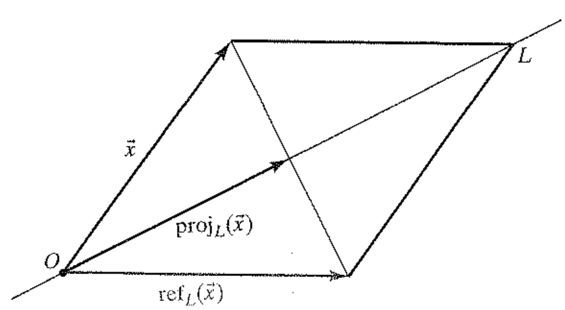


Figure 6

Orthogonal Projections and Reflections in Space

Although this section is mostly concerned with linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$, we will take a quick look at orthogonal projections and reflections in space, since this theory is analogous to the case of two dimensions.

Let L be a line in coordinate space, running through the origin. Any vector $\vec{x}$ in $\mathbb{R}^3$ can be written uniquely as $\vec{x} = \vec{x}^{\parallel} + \vec{x}^{\perp}$, where $\vec{x}^{\parallel}$ is parallel to L, and $\vec{x}^{\perp}$ is perpendicular to L. We define

$$\text{proj}_L(\vec{x}) = \vec{x}^{\parallel},$$

and we have the formula

$$\text{proj}_L(\vec{x}) = \vec{x}^{\parallel} = (\vec{x} \cdot \vec{u})\vec{u},$$

where $\vec{u}$ is a unit vector parallel to L. See Definition 2.2.1.

Let $L^{\perp} = V$ be the plane through the origin perpendicular to L; note that the vector $\vec{x}^{\perp}$ will be parallel to $L^{\perp} = V$. We can give formulas for the orthogonal projection onto V, as well as for the reflections about V and L, in terms of the orthogonal projection onto L:

$$\text{proj}_V(\vec{x}) = \vec{x} - \text{proj}_L(\vec{x}) = \vec{x} - (\vec{x} \cdot \vec{u})\vec{u},$$

$$\text{ref}_L(\vec{x}) = \text{proj}_L(\vec{x}) - \text{proj}_V(\vec{x}) = 2\text{proj}_L(\vec{x}) - \vec{x} = 2(\vec{x} \cdot \vec{u})\vec{u} - \vec{x}, \quad \text{and}$$

$$\text{ref}_V(\vec{x}) = \text{proj}_V(\vec{x}) - \text{proj}_L(\vec{x}) = -\text{ref}_L(\vec{x}) = \vec{x} - 2(\vec{x} \cdot \vec{u})\vec{u}.$$

See Figure 7, and compare with Definition 2.2.2.

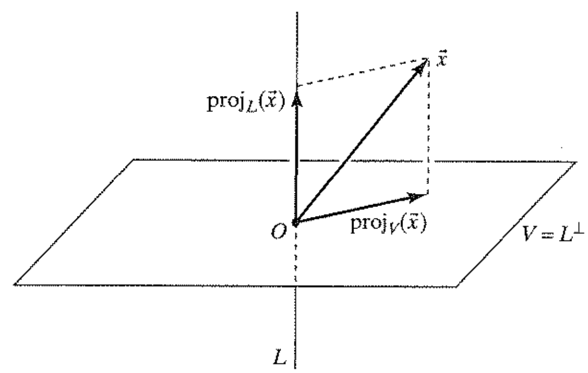


Figure 7

EXAMPLE 3 Let V be the plane defined by $2x_1 + x_2 - 2x_3 = 0$, and let $\vec{x} = \begin{bmatrix} 5 \\ 4 \\ -2 \end{bmatrix}$. Find $\text{ref}_V(\vec{x})$.

Solution

Note that the vector $\vec{v} = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ is perpendicular to plane V (the components of $\vec{v}$ are the coefficients of the variables in the given equation of the plane: 2, 1, and -2). Thus

$$\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v} = \frac{1}{3}\begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

is a unit vector perpendicular to V, and we can use the formula we derived earlier:

$$\begin{aligned}\operatorname{ref}_V(\vec{x}) = \vec{x} - 2(\vec{x}\cdot\vec{u})\vec{u} &= \begin{bmatrix}5\\4\\-2\end{bmatrix} - \frac{2}{9}\left(\begin{bmatrix}5\\4\\-2\end{bmatrix}\cdot\begin{bmatrix}2\\1\\-2\end{bmatrix}\right)\begin{bmatrix}2\\1\\-2\end{bmatrix}\\ &= \begin{bmatrix}5\\4\\-2\end{bmatrix} - 4\begin{bmatrix}2\\1\\-2\end{bmatrix}\\ &= \begin{bmatrix}5\\4\\-2\end{bmatrix} - \begin{bmatrix}8\\4\\-8\end{bmatrix}\\ &= \begin{bmatrix}-3\\0\\6\end{bmatrix}.\end{aligned}$$

■

Rotations

Consider the linear transformation T from $\mathbb{R}^2$ to $\mathbb{R}^2$ that rotates any vector $\vec{x}$ through a fixed angle θ in the counterclockwise direction,[5] as shown in Figure 8. Recall Example 2.1.5, where we studied a rotation through $\theta = \pi/2$.

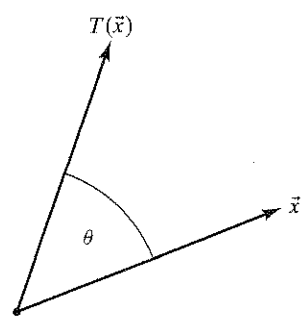


Figure 8

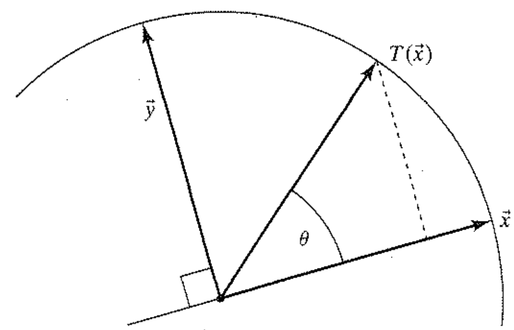


Figure 9

Now consider Figure 9, where we introduce the auxiliary vector $\vec{y}$, obtained by rotating $\vec{x}$ through $\pi/2$. From Example 2.1.5 we know that if $\vec{x} = \begin{bmatrix}x_1\\x_2\end{bmatrix}$, then $\vec{y} = \begin{bmatrix}-x_2\\x_1\end{bmatrix}$. Using basic trigonometry, we find that

$$\begin{aligned}T(\vec{x}) = (\cos\theta)\vec{x} + (\sin\theta)\vec{y} &= (\cos\theta)\begin{bmatrix}x_1\\x_2\end{bmatrix} + (\sin\theta)\begin{bmatrix}-x_2\\x_1\end{bmatrix}\\ &= \begin{bmatrix}(\cos\theta)x_1 - (\sin\theta)x_2\\(\sin\theta)x_1 + (\cos\theta)x_2\end{bmatrix}\\ &= \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}\\ &= \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\vec{x}.\end{aligned}$$

[5] We can define a rotation more formally in terms of the polar coordinates of $\vec{x}$. The length of $T(\vec{x})$ equals the length of $\vec{x}$, and the polar angle (or argument) of $T(\vec{x})$ exceeds the polar angle of $\vec{x}$ by θ.

This computation shows that a rotation through θ is indeed a linear transformation, with the matrix

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Theorem 2.2.3 Rotations

The matrix of a counterclockwise rotation in $\mathbb{R}^2$ through an angle θ is

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Note that this matrix is of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, where $a^2 + b^2 = 1$. Conversely, any matrix of this form represents a rotation. ■

EXAMPLE 4 The matrix of a counterclockwise rotation through $\pi/6$ (or 30°) is

$$\begin{bmatrix} \cos(\pi/6) & -\sin(\pi/6) \\ \sin(\pi/6) & \cos(\pi/6) \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \sqrt{3} & -1 \\ 1 & \sqrt{3} \end{bmatrix}.$$ ■

Rotations Combined with a Scaling

EXAMPLE 5 Examine how the linear transformation

$$T(\vec{x}) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}\vec{x}$$

affects our standard letter L. Here a and b are arbitrary constants.

Solution

Figure 10 suggests that T represents a *rotation combined with a scaling*. Think polar coordinates: This is a rotation through the phase angle θ of vector $\begin{bmatrix} a \\ b \end{bmatrix}$, combined with a scaling by the magnitude $r = \sqrt{a^2 + b^2}$ of vector $\begin{bmatrix} a \\ b \end{bmatrix}$. To verify this claim algebraically, we can write the vector $\begin{bmatrix} a \\ b \end{bmatrix}$ in polar coordinates, as

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} r\cos\theta \\ r\sin\theta \end{bmatrix},$$

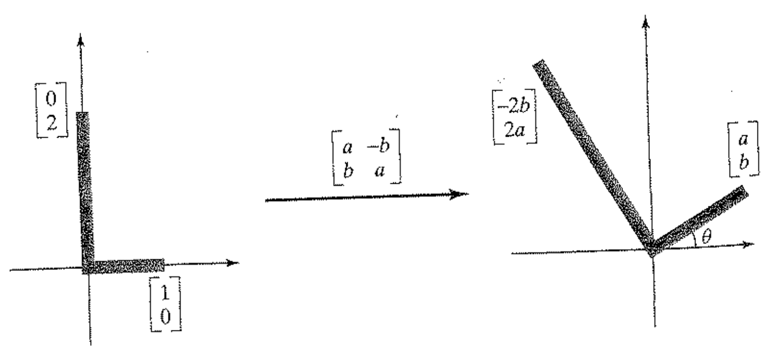


Figure 10

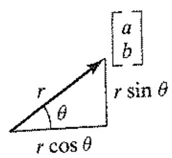


Figure 11

as illustrated in Figure 11. Then

$$\begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} r\cos\theta & -r\sin\theta \\ r\sin\theta & r\cos\theta \end{bmatrix} = r\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

It turns out that matrix $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ is a scalar multiple of a rotation matrix, as claimed. ∎

Theorem 2.2.4 **Rotations combined with a scaling**

A matrix of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ represents a rotation combined with a scaling.

More precisely, if r and θ are the polar coordinates of vector $\begin{bmatrix} a \\ b \end{bmatrix}$, then $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ represents a rotation through θ combined with a scaling by r. ∎

Shears

We will introduce shears by means of some simple experiments involving a ruler and a deck of cards.[6]

In the first experiment, we place the deck of cards on the ruler, as shown in Figure 12. Note that the 2 of diamonds is placed on one of the short edges of the ruler. That edge will stay in place throughout the experiment. Now we lift the other short edge of the ruler up, keeping the cards in vertical position at all times. The cards will slide up, being "fanned out," without any horizontal displacement.

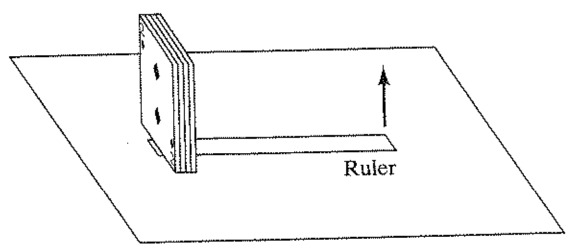


Figure 12

Figure 13 shows a side view of this transformation. The origin represents the ruler's short edge that is staying in place.

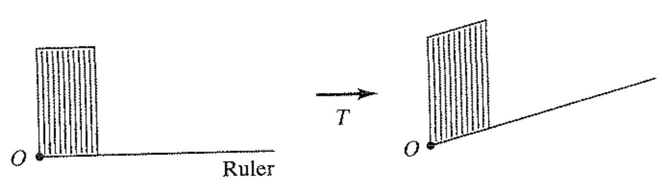


Figure 13

Such a transformation T is called a *vertical shear*. If we focus on the side view only, we have a vertical shear in $\mathbb{R}^2$ (although in reality the experiment takes place in space).

[6]Two hints for instructors:

- Use several decks of cards for dramatic effect.
- Hold the decks together with a rubber band to avoid embarrassing accidents.

Now let's draw a vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ on the side of our deck of cards, and let's find a formula for the sheared vector $T(\vec{x})$, using Figure 14 as a guide. Here, k denotes the slope of the ruler after the transformation:

$$T(\vec{x}) = T\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_1 \\ kx_1 + x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}\vec{x}.$$

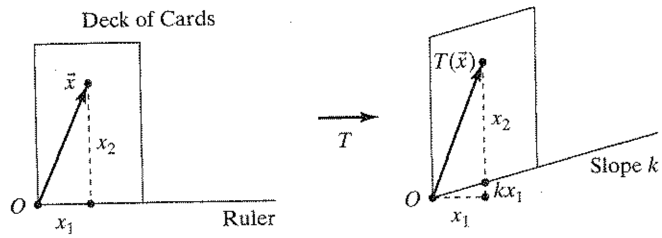


Figure 14

We find that the matrix of a vertical shear is of the form $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$, where k is an arbitrary constant.

Horizontal shears are defined analogously; consider Figure 15.

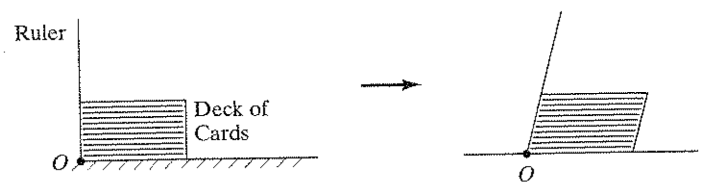


Figure 15

We leave it as an exercise to the reader to verify that the matrix of a horizontal shear is of the form $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$. Take another look at part (e) of Example 1.

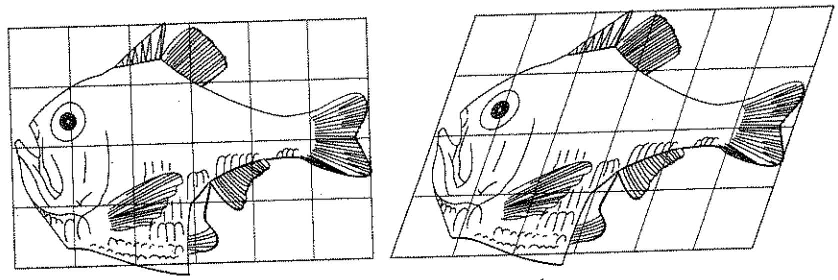
Oblique shears are far less important in applications, and we will not consider them in this introductory text.

Theorem 2.2.5 **Horizontal and vertical shears**

The matrix of a *horizontal shear* is of the form $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$, and the matrix of a *vertical shear* is of the form $\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$, where k is an arbitrary constant. ■

The Scottish scholar d'Arcy Thompson showed how the shapes of related species of plants and animals can often be transformed into one another, using linear as well as nonlinear transformations.[7] In Figure 16 he uses a horizontal shear to transform the shape of one species of fish into another.

[7]Thompson, d'Arcy W., *On Growth and Form*, Cambridge University Press, 1917. P. B. Medawar calls this "the finest work of literature in all the annals of science that have been recorded in the English tongue."



Argyropelecus olfersi. *Sternoptyx diaphana.*

Figure 16

EXERCISES 2.2

GOAL *Use the matrices of orthogonal projections, reflections, and rotations. Apply the definitions of shears, orthogonal projections, and reflections.*

1. Sketch the image of the standard L under the linear transformation

$$T(\vec{x}) = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \vec{x}.$$

(See Example 1.)

2. Find the matrix of a rotation through an angle of 60° in the counterclockwise direction.

3. Consider a linear transformation T from $\mathbb{R}^2$ to $\mathbb{R}^3$. Use $T(\vec{e}_1)$ and $T(\vec{e}_2)$ to describe the image of the unit square geometrically.

4. Interpret the following linear transformation geometrically:

$$T(\vec{x}) = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \vec{x}.$$

5. The matrix

$$\begin{bmatrix} -0.8 & -0.6 \\ 0.6 & -0.8 \end{bmatrix}$$

represents a rotation. Find the angle of rotation (in radians).

6. Let L be the line in $\mathbb{R}^3$ that consists of all scalar multiples of the vector $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$. Find the orthogonal projection of the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ onto L.

7. Let L be the line in $\mathbb{R}^3$ that consists of all scalar multiples of $\begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$. Find the reflection of the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ about the line L.

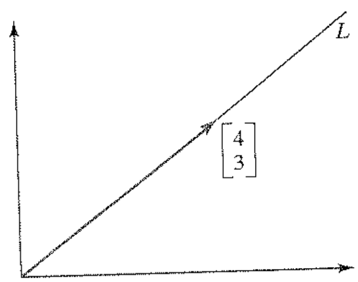
8. Interpret the following linear transformation geometrically:

$$T(\vec{x}) = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \vec{x}.$$

9. Interpret the following linear transformation geometrically:

$$T(\vec{x}) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \vec{x}.$$

10. Find the matrix of the orthogonal projection onto the line L in $\mathbb{R}^2$ shown in the accompanying figure:



11. Refer to Exercise 10. Find the matrix of the reflection about the line L.

12. Consider a line L in the plane, running through the origin. If $\vec{w} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix}$ is a nonzero vector parallel to L, show that the matrix of $\text{proj}_L(\vec{x})$ is

$$\frac{1}{w_1^2 + w_2^2} \begin{bmatrix} w_1^2 & w_1 w_2 \\ w_1 w_2 & w_2^2 \end{bmatrix}.$$

13. Suppose a line L in $\mathbb{R}^2$ contains the unit vector

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}.$$

Find the matrix A of the linear transformation $T(\vec{x}) = \text{ref}_L(\vec{x})$. Give the entries of A in terms of u_1 and u_2. Show that A is of the form $\begin{bmatrix} a & b \\ b & -a \end{bmatrix}$, where $a^2 + b^2 = 1$.

14. Suppose a line L in $\mathbb{R}^3$ contains the unit vector

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}.$$

a. Find the matrix A of the linear transformation $T(\vec{x}) = \text{proj}_L(\vec{x})$. Give the entries of A in terms of the components u_1, u_2, u_3 of $\vec{u}$.

b. What is the sum of the diagonal entries of the matrix A you found in part (a)?

15. Suppose a line L in $\mathbb{R}^3$ contains the unit vector

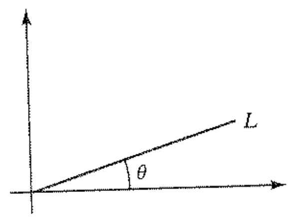
$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}.$$

Find the matrix A of the linear transformation $T(\vec{x}) = \text{ref}_L(\vec{x})$. Give the entries of A in terms of the components u_1, u_2, u_3 of $\vec{u}$.

16. Let $T(\vec{x}) = \text{ref}_L(\vec{x})$ be the reflection about the line L in $\mathbb{R}^2$ shown in the accompanying figure.

a. Draw sketches to illustrate that T is linear.

b. Find the matrix of T in terms of θ.



17. Consider a matrix A of the form $A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$, where $a^2 + b^2 = 1$. Find two nonzero perpendicular vectors $\vec{v}$ and $\vec{w}$ such that $A\vec{v} = \vec{v}$ and $A\vec{w} = -\vec{w}$ (write the entries of $\vec{v}$ and $\vec{w}$ in terms of a and b). Conclude that $T(\vec{x}) = A\vec{x}$ represents the reflection about the line L spanned by $\vec{v}$.

18. The linear transformation $T(\vec{x}) = \begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix} \vec{x}$ is a reflection about a line L (see Exercise 17). Find the equation of line L (in the form $y = mx$).

Find the matrices of the linear transformations from $\mathbb{R}^3$ to $\mathbb{R}^3$ given in Exercises 19 through 23. Some of these transformations have not been formally defined in the text. Use common sense. You may assume that all these transformations are linear.

19. The orthogonal projection onto the x–y-plane.

20. The reflection about the x–z-plane.

21. The rotation about the z-axis through an angle of $\pi/2$, counterclockwise as viewed from the positive z-axis.

22. The rotation about the y-axis through an angle θ, counterclockwise as viewed from the positive y-axis.

23. The reflection about the plane $y = z$.

24. Rotations and reflections have two remarkable properties: They preserve the length of vectors and the angle between vectors. (Draw figures illustrating these properties.) We will show that, conversely, any linear transformation T from $\mathbb{R}^2$ to $\mathbb{R}^2$ that preserves length and angles is either a rotation or a reflection (about a line).

a. Show that if $T(\vec{x}) = A\vec{x}$ preserves length and angles, then the two column vectors $\vec{v}$ and $\vec{w}$ of A must be perpendicular unit vectors.

b. Write the first column vector of A as $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$; note that $a^2 + b^2 = 1$, since $\vec{v}$ is a unit vector. Show that for a given $\vec{v}$ there are two possibilities for $\vec{w}$, the second column vector of A. Draw a sketch showing $\vec{v}$ and the two possible vectors $\vec{w}$. Write the components of $\vec{w}$ in terms of a and b.

c. Show that if a linear transformation T from $\mathbb{R}^2$ to $\mathbb{R}^2$ preserves length and angles, then T is either a rotation or a reflection (about a line). See Exercise 17.

25. Find the inverse of the matrix $\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$, where k is an arbitrary constant. Interpret your result geometrically.

* **26. a.** Find the scaling matrix A that transforms $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ into $\begin{bmatrix} 8 \\ -4 \end{bmatrix}$.

b. Find the orthogonal projection matrix B that transforms $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ into $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$.

c. Find the rotation matrix C that transforms $\begin{bmatrix} 0 \\ 5 \end{bmatrix}$ into $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$.

d. Find the shear matrix D that transforms $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ into $\begin{bmatrix} 7 \\ 3 \end{bmatrix}$.

e. Find the reflection matrix E that transforms $\begin{bmatrix} 7 \\ 1 \end{bmatrix}$ into $\begin{bmatrix} -5 \\ 5 \end{bmatrix}$.

27. Consider the matrices A through E below.

$$A = \begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix},$$

$$C = \begin{bmatrix} 0.36 & -0.48 \\ -0.48 & 0.64 \end{bmatrix}, \quad D = \begin{bmatrix} -0.8 & 0.6 \\ -0.6 & -0.8 \end{bmatrix},$$

$$E = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

Fill in the blanks in the sentences below.
We are told that there is a solution in each case.
Matrix ______ represents a scaling.
Matrix ______ represents an orthogonal projection.
Matrix ______ represents a shear.
Matrix ______ represents a reflection.
Matrix ______ represents a rotation.

28. Each of the linear transformations in parts (a) through (e) corresponds to one (and only one) of the matrices A through J. Match them up.

a. Scaling **b.** Shear **c.** Rotation
d. Orthogonal Projection **e.** Reflection

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} \quad C = \begin{bmatrix} -0.6 & 0.8 \\ -0.8 & -0.6 \end{bmatrix}$$

$$D = \begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} \quad E = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \quad F = \begin{bmatrix} 0.6 & 0.8 \\ 0.8 & -0.6 \end{bmatrix}$$

$$G = \begin{bmatrix} 0.6 & 0.6 \\ 0.8 & 0.8 \end{bmatrix} \quad H = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \quad I = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$J = \begin{bmatrix} 0.8 & -0.6 \\ 0.6 & -0.8 \end{bmatrix}$$

29. Let T be a function from $\mathbb{R}^m$ to $\mathbb{R}^n$, and let L be a function from $\mathbb{R}^n$ to $\mathbb{R}^m$. Suppose that $L(T(\vec{x})) = \vec{x}$ for all $\vec{x}$ in $\mathbb{R}^m$ and $T(L(\vec{y})) = \vec{y}$ for all $\vec{y}$ in $\mathbb{R}^n$. If T is a linear transformation, show that L is linear as well. [*Hint:* $\vec{v} + \vec{w} = T(L(\vec{v})) + T(L(\vec{w})) = T(L(\vec{v}) + L(\vec{w}))$ since T is linear. Now apply L on both sides.]

30. Find a nonzero 2×2 matrix A such that $A\vec{x}$ is parallel to the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, for all $\vec{x}$ in $\mathbb{R}^2$.

31. Find a nonzero 3×3 matrix A such that $A\vec{x}$ is perpendicular to $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, for all $\vec{x}$ in $\mathbb{R}^3$.

32. Consider the rotation matrix $D = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$ and the vector $\vec{v} = \begin{bmatrix} \cos\beta \\ \sin\beta \end{bmatrix}$, where α and β are arbitrary angles.

a. Draw a sketch to explain why $D\vec{v} = \begin{bmatrix} \cos(\alpha+\beta) \\ \sin(\alpha+\beta) \end{bmatrix}$.

b. Compute $D\vec{v}$. Use the result to derive the addition theorems for sine and cosine:

$$\cos(\alpha+\beta) = \ldots, \quad \sin(\alpha+\beta) = \ldots.$$

33. Consider two nonparallel lines L_1 and L_2 in $\mathbb{R}^2$. Explain why a vector $\vec{v}$ in $\mathbb{R}^2$ can be expressed uniquely as

$$\vec{v} = \vec{v}_1 + \vec{v}_2,$$

where $\vec{v}_1$ is on L_1 and $\vec{v}_2$ on L_2. Draw a sketch. The transformation $T(\vec{v}) = \vec{v}_1$ is called the *projection onto* L_1 *along* L_2. Show algebraically that T is linear.

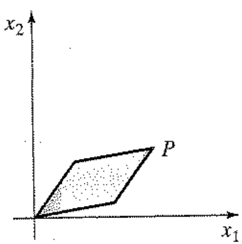
34. One of the five given matrices represents an orthogonal projection onto a line and another represents a reflection about a line. Identify both and briefly justify your choice.

$$A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}, \quad B = \frac{1}{3}\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

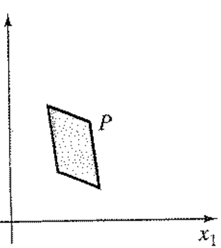
$$C = \frac{1}{3}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}, \quad D = -\frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix},$$

$$E = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}$$

35. Let T be an invertible linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$. Let P be a parallelogram in $\mathbb{R}^2$ with one vertex at the origin. Is the image of P a parallelogram as well? Explain. Draw a sketch of the image.



36. Let T be an invertible linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$. Let P be a parallelogram in $\mathbb{R}^2$. Is the image of P a parallelogram as well? Explain.



37. The *trace* of a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the sum $a + d$ of its diagonal entries. What can you say about the trace of a 2×2 matrix that represents a(n)

a. orthogonal projection **b.** reflection about a line
c. rotation **d.** (horizontal or vertical) shear.

In three cases, give the exact value of the trace, and in one case, give an interval of possible values.

38. The *determinant* of a matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad - bc$ (we have seen this quantity in Exercise 2.1.13 already). Find the determinant of a matrix that represents a(n)

a. orthogonal projection **b.** reflection about a line

c. rotation **d.** (horizontal or vertical) shear.

What do your answers tell you about the invertibility of these matrices?

39. Describe each of the linear transformations defined by the matrices in parts (a) through (c) geometrically, as a well-known transformation combined with a scaling. Give the scaling factor in each case.

a. $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ **b.** $\begin{bmatrix} 3 & 0 \\ -1 & 3 \end{bmatrix}$

c. $\begin{bmatrix} 3 & 4 \\ 4 & -3 \end{bmatrix}$

40. Let P and Q be two perpendicular lines in $\mathbb{R}^2$. For a vector $\vec{x}$ in $\mathbb{R}^2$, what is $\text{proj}_P(\vec{x}) + \text{proj}_Q(\vec{x})$? Give your answer in terms of $\vec{x}$. Draw a sketch to justify your answer.

41. Let P and Q be two perpendicular lines in $\mathbb{R}^2$. For a vector $\vec{x}$ in $\mathbb{R}^2$, what is the relationship between $\text{ref}_P(\vec{x})$ and $\text{ref}_Q(\vec{x})$? Draw a sketch to justify your answer.

42. Let $T(\vec{x}) = \text{proj}_L(\vec{x})$ be the orthogonal projection onto a line in $\mathbb{R}^2$. What is the relationship between $T(\vec{x})$ and $T(T(\vec{x}))$? Justify your answer carefully.

43. Use the formula derived in Exercise 2.1.13 to find the inverse of the rotation matrix

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Interpret the linear transformation defined by A^{-1} geometrically. Explain.

44. A nonzero matrix of the form $A = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ represents a rotation combined with a scaling. Use the formula derived in Exercise 2.1.13 to find the inverse of A. Interpret the linear transformation defined by A^{-1} geometrically. Explain.

45. A matrix of the form $A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$, where $a^2+b^2 = 1$, represents a reflection about a line (see Exercise 17). Use the formula derived in Exercise 2.1.13 to find the inverse of A. Explain.

46. A nonzero matrix of the form $A = \begin{bmatrix} a & b \\ b & -a \end{bmatrix}$ represents a reflection about a line L combined with a scaling. (Why? What is the scaling factor?) Use the formula derived in Exercise 2.1.13 to find the inverse of A. Interpret the linear transformation defined by A^{-1} geometrically. Explain.

47. Let T be a linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$. Consider the function

$$f(t) = \left(T\begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix}\right) \cdot \left(T\begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix}\right),$$

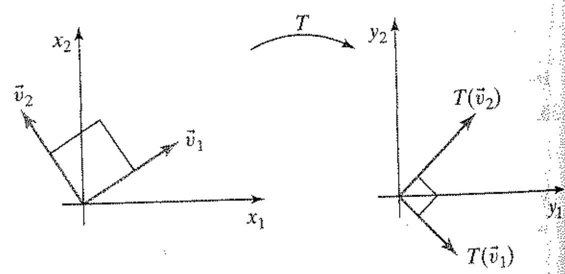
from $\mathbb{R}$ to $\mathbb{R}$. Show the following:

a. The function $f(t)$ is continuous. You may take for granted that the functions $\sin(t)$ and $\cos(t)$ are continuous, and also that sums and products of continuous functions are continuous.

b. $f(\pi/2) = -f(0)$.

c. There exists a number c between 0 and $\pi/2$ such that $f(c) = 0$. Use the intermediate value theorem of calculus, which tells us the following: If a function $g(t)$ is continuous for $a \le t \le b$, and L is a number between $g(a)$ and $g(b)$, then there exists at least one number c between a and b such that $g(c) = L$.

d. There exist two perpendicular unit vectors $\vec{v}_1$ and $\vec{v}_2$ in $\mathbb{R}^2$ such that the vectors $T(\vec{v}_1)$ and $T(\vec{v}_2)$ are perpendicular as well. See the accompanying figure. (Compare with Theorem 8.3.3 for a generalization.)



48. Refer to Exercise 47. Consider the linear transformation

$$T(\vec{x}) = \begin{bmatrix} 0 & 4 \\ 5 & -3 \end{bmatrix}\vec{x}.$$

Find the function $f(t)$ defined in Exercise 47, graph it (using technology), and find a number c between 0 and $\pi/2$ such that $f(c) = 0$. Use your answer to find two perpendicular unit vectors $\vec{v}_1$ and $\vec{v}_2$ such that $T(\vec{v}_1)$ and $T(\vec{v}_2)$ are perpendicular. Draw a sketch.

49. Sketch the image of the unit circle under the linear transformation

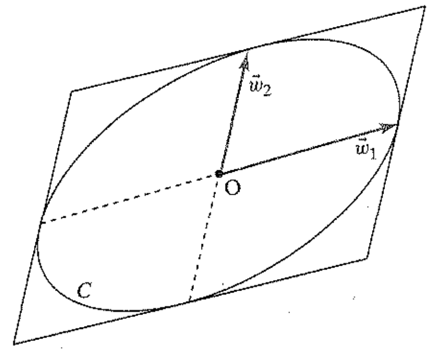
$$T(\vec{x}) = \begin{bmatrix} 5 & 0 \\ 0 & 2 \end{bmatrix}\vec{x}.$$

50. Let T be an invertible linear transformation from $\mathbb{R}^2$ to $\mathbb{R}^2$. Show that the image of the unit circle is an ellipse centered at the origin.[8] [*Hint*: Consider two perpendicular unit vectors $\vec{v}_1$ and $\vec{v}_2$ such that $T(\vec{v}_1)$ and $T(\vec{v}_2)$ are perpendicular.] (See Exercise 47d.) The unit circle consists of all vectors of the form

$$\vec{v} = \cos(t)\vec{v}_1 + \sin(t)\vec{v}_2,$$

where t is a parameter.

51. Let $\vec{w}_1$ and $\vec{w}_2$ be two nonparallel vectors in $\mathbb{R}^2$. Consider the curve C in $\mathbb{R}^2$ that consists of all vectors of the form $\cos(t)\vec{w}_1 + \sin(t)\vec{w}_2$, where t is a parameter. Show that C is an ellipse. (*Hint:* You can interpret C as the image of the unit circle under a suitable linear transformation; then use Exercise 50.)



52. Consider an invertible linear transformation T from $\mathbb{R}^2$ to $\mathbb{R}^2$. Let C be an ellipse in $\mathbb{R}^2$. Show that the image of C under T is an ellipse as well. (*Hint*: Use the result of Exercise 51.)

2.3 Matrix Products

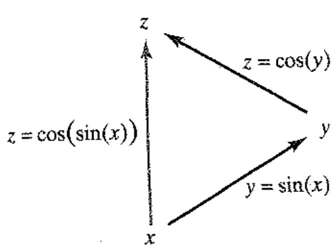


Figure 1

Recall the *composition* of two functions: The composite of the functions $y = \sin(x)$ and $z = \cos(y)$ is $z = \cos(\sin(x))$, as illustrated in Figure 1.

Similarly, we can compose two linear transformations.

To understand this concept, let's return to the coding example discussed in Section 2.1. Recall that the position $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ of your boat is encoded and that you radio the encoded position $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ to Marseille. The coding transformation is

$$\vec{y} = A\vec{x}, \quad \text{with} \quad A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}.$$

In Section 2.1, we left out one detail: Your position is radioed on to Paris, as you would expect in a centrally governed country such as France. Before broadcasting to Paris, the position $\vec{y}$ is again encoded, using the linear transformation

$$\vec{z} = B\vec{y}, \quad \text{with} \quad B = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$$

[8] An ellipse in $\mathbb{R}^2$ centered at the origin may be defined as a curve that can be parametrized as

$$\cos(t)\vec{w}_1 + \sin(t)\vec{w}_2,$$

for two perpendicular vectors $\vec{w}_1$ and $\vec{w}_2$. Suppose the length of $\vec{w}_1$ exceeds the length of $\vec{w}_2$. Then we call the vectors $\pm\vec{w}_1$ the semimajor axes of the ellipse and $\pm\vec{w}_2$ the semiminor axes.

Convention: All ellipses considered in this text are centered at the origin unless stated otherwise.