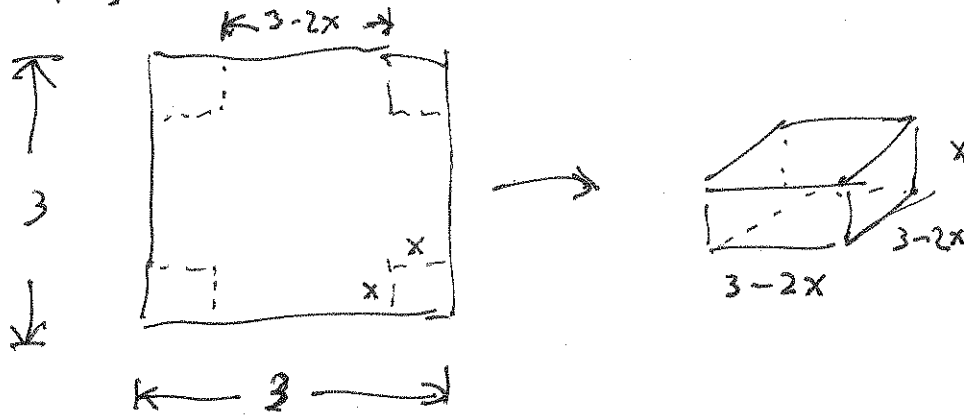


**Optimization**

Example 1: A box with an open top is to be constructed from a square piece of cardboard, 3 feet wide, by cutting out a square from each of the four corners and bending up the sides. Find the largest volume that such a box can have.

(1) Draw a pic.



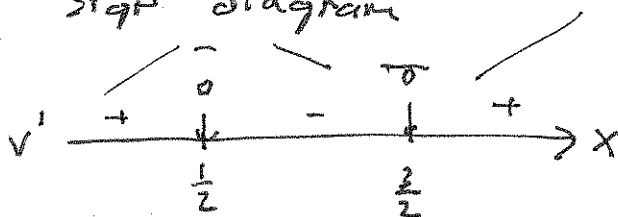
(2) set up an eqn.

$$\begin{aligned}
 v(x) &= x(3-2x)^2 \\
 &= x(9-12x+4x^2) \\
 &= 4x^3 - 12x^2 + 9x
 \end{aligned}$$

(3) Take the deriv.

$$\begin{aligned}
 v'(x) &= 12x^2 - 24x + 9 \\
 &= 3(4x^2 - 8x + 3) \\
 &= 3(2x - 1)(2x - 3)
 \end{aligned}$$

(4) sign diagram

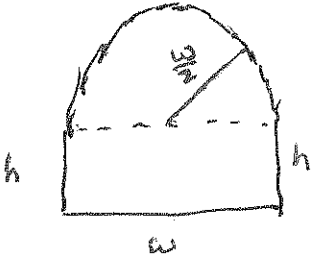


max at  $(\frac{1}{2}, 2)$

(5) The max volume is  $2 \text{ ft}^3$ .

Example 2: A Norman window has the shape of a rectangle surmounted by a semicircle. If the perimeter of the window is 30 feet, find the dimensions of the window so that the greatest possible amount of light is admitted.

(1) Draw a pic



(2) Set up an eqn.

$$\text{perimeter: } 2h + w + \pi \cdot \frac{w}{2} = 30 \Rightarrow h = \frac{30 - \frac{\pi \cdot w}{2} - w}{2}$$

$$\text{area: } A = hw + \frac{\pi \cdot \left(\frac{w}{2}\right)^2}{2}$$

$$= \frac{30 - \frac{\pi \cdot w}{2} - w}{2} \cdot w + \frac{\pi w^2}{8}$$

$$= 15w - \frac{\pi w^2}{4} - \frac{w^2}{2} + \frac{\pi w^2}{8}$$

$$= 15w + w^2 \left( -\frac{\pi}{4} - \frac{1}{2} + \frac{\pi}{8} \right)$$

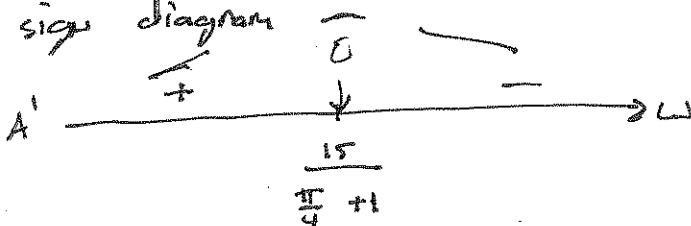
$$= 15w + \left( -\frac{\pi}{8} - \frac{1}{2} \right) w^2$$

(3) Take deriv.

$$A' = 15 - 2 \left( \frac{\pi}{8} + \frac{1}{2} \right) w$$

$$\text{zero @ } w = \frac{15}{2 \left( \frac{\pi}{8} + \frac{1}{2} \right)}$$

(4) sign diagram

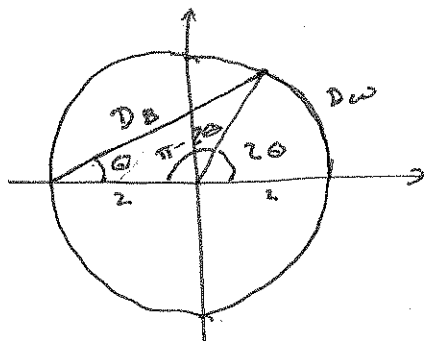


$$\text{max @ } w = \frac{15}{\frac{\pi}{4} + 1} \approx 8.40$$

(5) Answer.

The optimal window has width 8.4 ft and height of 4.2 ft.

Example 3: A woman is at the boat launch of a circular lake with radius 2 miles that has a sidewalk all the way around it. She wants to arrive at the point diametrically opposite the launch (on the other side of the lake) in the shortest possible time. She can walk at the rate of 4 mph and row a boat at 2mph. How should she proceed?



$$D_w = 2 \cdot 2\theta$$

$$D_B = \sqrt{8 - 8 \cos(\pi - 2\theta)}$$

$$= 2\sqrt{2 - 2 \cos(\pi - 2\theta)}$$

$$\text{and } 0 \leq \theta \leq \frac{\pi}{2}$$

$$\text{so } T(\theta) = \frac{D_w}{4} + \frac{D_B}{2}$$

$$= \theta + \sqrt{2 - 2 \cos(\pi - 2\theta)}$$

$$\Rightarrow T'(\theta) = 1 + \frac{+2 \sin(\pi - 2\theta)(-2)}{2\sqrt{2 - 2 \cos(\pi - 2\theta)}}$$

$$= 1 - \frac{2 \sin(\pi - 2\theta)}{\sqrt{2 - 2 \cos(\pi - 2\theta)}}$$

and solve  $T'(\theta) = 0$

$$\frac{1}{2} = \frac{\sin(\pi - 2\theta)}{\sqrt{2 - 2 \cos(\pi - 2\theta)}}$$

$$\Rightarrow \frac{1}{4} = \frac{\sin^2(\pi - 2\theta)}{2 - 2 \cos(\pi - 2\theta)}$$

$$\Rightarrow \frac{1}{4}(2 - 2 \cos(\pi - 2\theta)) = 1 - \cos^2(\pi - 2\theta)$$

$$\Rightarrow \cos^2(\pi - 2\theta) - \frac{1}{2} \cos(\pi - 2\theta) - \frac{1}{2} = 0$$

$$\Rightarrow 2c^2 - c - 1 = 0 \text{ where } c = \cos(\pi - 2\theta)$$

$$\Rightarrow (2c + 1)(c - 1) = 0$$

$$\Rightarrow c = -\frac{1}{2} \text{ or } c = 1$$

$$\text{so } \cos(\pi - 2\theta) = 1$$

$$\Rightarrow \pi - 2\theta = 0 \Rightarrow \theta = \frac{\pi}{2}$$

or

$$\pi - 2\theta = 2\pi \Rightarrow \theta = -\frac{\pi}{2}$$

(outside domain)

$$\text{or } \cos(\pi - 2\theta) = -\frac{1}{2}$$

$$\Rightarrow \pi - 2\theta = \frac{2\pi}{3} \Rightarrow \theta = \frac{\pi}{6}$$

or

$$\pi - 2\theta = \frac{4\pi}{3} \Rightarrow \theta = -\frac{\pi}{6}$$

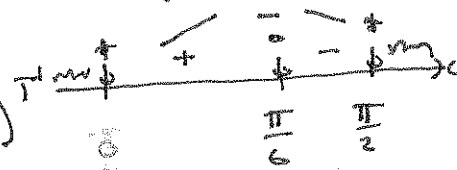
(outside domain)

But  $T'$  is undefined when  $\cos(\pi - 2\theta) = 1$  or  $\theta = \pm \frac{\pi}{2}$

Did we pick up other extraneous solutions?

$$T'(\frac{\pi}{6}) = 0 \text{ \& } T'(-\frac{\pi}{6}) = 2$$

(extra soln)



There is a max time when  $\theta = \frac{\pi}{6}$

hmmm,

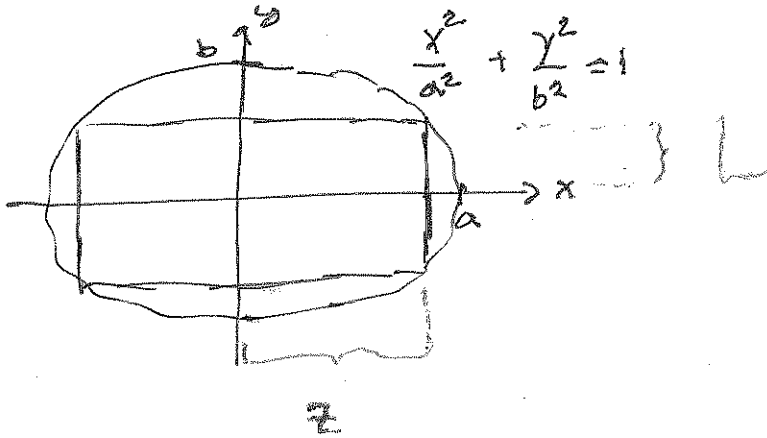
check endpoints.

$$T(0) = 2 \text{ hrs; } T(\frac{\pi}{2}) = 1.5$$

She should walk the whole way.

Example 4: Find the area of the largest rectangle that can be inscribed in the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(1) draw a pic.



(2) set up eqn.

$$\text{Area} = (2z)(2L) = 4zL$$

$$\text{and } L = \pm \frac{b}{a} \sqrt{a^2 - z^2}$$

$$\text{so } A = 4z \cdot \frac{b}{a} \sqrt{a^2 - z^2}$$

(3) differentiate.

$$A' = \frac{4b}{a} \left( 1 \cdot \sqrt{a^2 - z^2} + z \cdot \frac{-2z}{2\sqrt{a^2 - z^2}} \right)$$

$$= \frac{4b}{a} \cdot \frac{(a^2 - z^2) - z^2}{\sqrt{a^2 - z^2}}$$

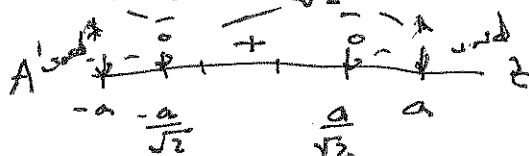
$$= \frac{4b}{a} \cdot \frac{a^2 - 2z^2}{\sqrt{a^2 - z^2}}$$

(4) find zeros ( $A' = 0$ ).

$$\text{solve } a^2 - 2z^2 = 0$$

$$\Rightarrow \frac{a^2}{2} = z^2$$

$$\Rightarrow z = \pm \frac{a}{\sqrt{2}}$$



(5) Maximize area

$$z = \frac{a}{\sqrt{2}}$$

$$A = 4 \cdot \frac{a}{\sqrt{2}} \cdot \frac{b}{a} \sqrt{a^2 - \left(\frac{a}{\sqrt{2}}\right)^2}$$

$$= \frac{4b}{\sqrt{2}} \sqrt{\frac{a^2}{2}}$$

$$= \frac{4ab}{2}$$

$$= 2ab$$

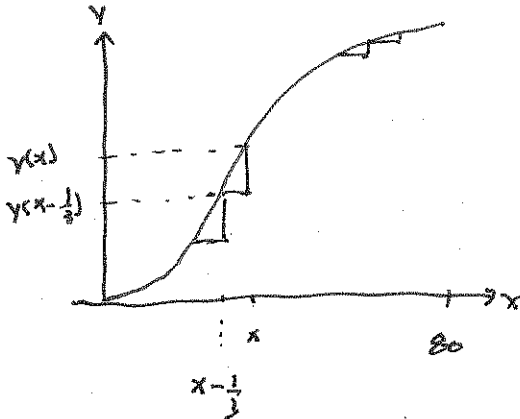
# Solution 1: using a difference

Example 5: A contractor is engaged to build steps up the slope of a hill that has the shape of the graph

$$y = \frac{x^2(120-x)}{6400}, 0 \leq x \leq 80 \text{ with } x \text{ in meters. What is the maximum vertical rise of a stair if each stair}$$

has a horizontal length of one-third meter?

(1) pic



stair hc for  $h(x) = y(x) - y(x - \frac{1}{3})$

this is valid for  $x = \frac{1}{3}, \frac{2}{3}, \dots, 80$

(2) eqn:

$$h(x) = y(x) - y(x - \frac{1}{3})$$

$$= \frac{1}{6400} \left( -x^2 + \frac{2169}{27}x - \frac{361}{27} \right)$$

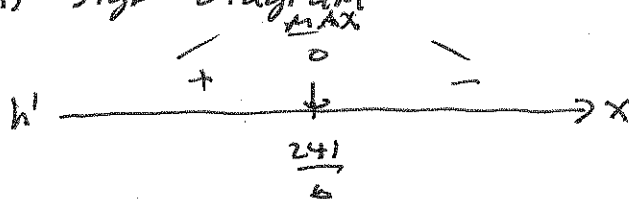
(I did this in Mathematica)

(3) differentiate

$$h'(x) = \frac{1}{6400} \left( -2x + \frac{2169}{27} \right)$$

$$h'(x) = 0 \text{ when } x = \frac{241}{6}$$

(4) sign diagram



The max is at  $\frac{241}{6}$  which is not in the domain. So we check the nearest values to or  $40\frac{1}{3}$ .

$$\rightarrow h(40) = h(40\frac{1}{3}) = \frac{43199}{172800} \approx 0.249994$$

(5) conclusion:

The max stair height is about  $\frac{1}{4}$  m.

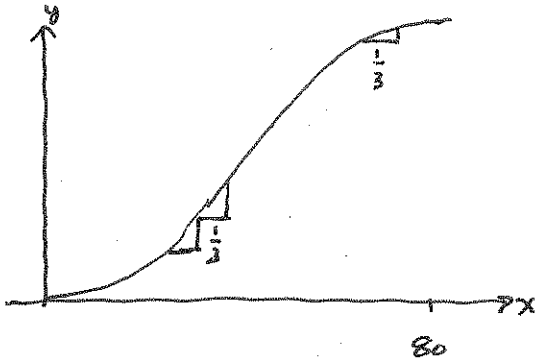
## Solution 2: using linearization

Example 5: A contractor is engaged to build steps up the slope of a hill that has the shape of the graph

$$y = \frac{x^2(120-x)}{6400}, 0 \leq x \leq 80 \text{ with } x \text{ in meters. What is the maximum vertical rise of a stair if each stair}$$

has a horizontal length of one-third meter?

(1) pic



observe: the stair ht  
is more where  
the hill is steepest.

$y$ : hill ht

$y'$ : hill steepness

$y''$ : find max hill steepness.

(2) eqn.

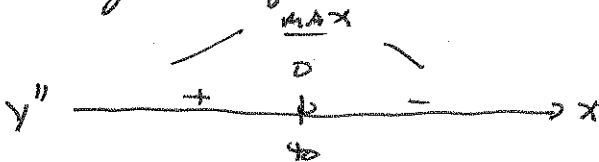
$$y = \frac{1}{6400} (120x^2 - x^3)$$

$$y' = \frac{1}{6400} (240x - 3x^2)$$

(3) differentiate

$$y'' = \frac{1}{6400} (240 - 6x)$$

(4) sign diagram



$$\text{max slope} = y'(40) = \frac{3}{4}$$

The run is  $\frac{1}{3}$  and so the rise =  $\frac{1}{4}$

(5) conclusion:

The max stair height is  $\frac{1}{4}$  m.

Example 6: An 8-billion-bushel corn crop brings a price of \$2.40/bu. A commodity broker uses the rule of thumb: If the crop is reduced by  $x$  percent, then the price increases by  $10x$  cents. Which crop size results in maximum revenue and what is the price per bushel?

(1) table

	q	price	revenue
	8	2.40	$8(2.4)$
-1%	7.92	2.50	$(7.92)(2.5)$
-2%	7.84	2.60	$\vdots$
-3%	7.76	2.70	$\vdots$
$\vdots$			
-x%	$8 - .08x$	$2.40 + .10x$	$R(x) = (8 - .08x)(2.4 + .10x)$

(2) set up eqn.

$$R(x) = (8 - 0.08x)(2.4 + 0.10x)$$

(3) deriv.

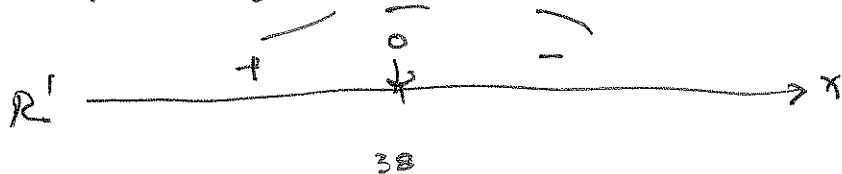
$$R'(x) = -0.08(2.4 + .1x) + .1(8 - 0.08x)$$

$$= -.192 - .008x + .8 - .008x$$

$$= .608 - 0.016x$$

$$R' = 0 \text{ when } x = 38$$

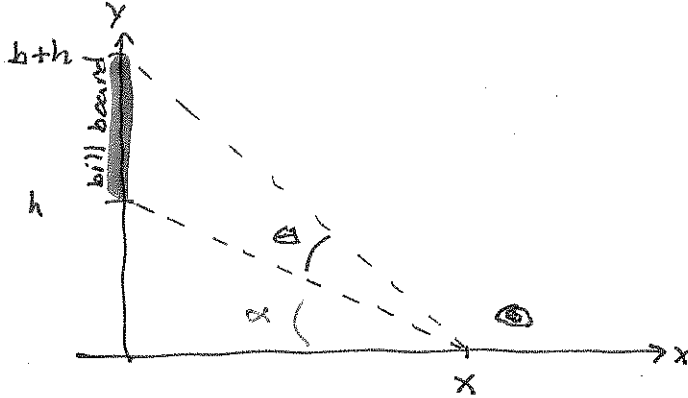
(4) sign diagram  $\overbrace{\text{max}}$



(5) our max rev. is when  
 $8 - 38(.08) = 4.96$  bil. bushels of  
 corn are produced. Each  
 bushel is sold for \$6.20/bu.

**Example 7:** A billboard of height  $b$  is mounted on the side of a building with its bottom edge at a distance  $h$  from the street below. At what distance  $x$  should an observer stand from the wall to maximize the angle of observation?

(1) Draw picture



(2) set up eqn.

$$\tan(\alpha) = \frac{h}{x} \quad \text{and} \quad \tan(\alpha + \theta) = \frac{h+b}{x}$$

$$\Rightarrow \alpha = \arctan\left(\frac{h}{x}\right) \quad \text{and} \quad \alpha + \theta = \arctan\left(\frac{h+b}{x}\right)$$

$$\Rightarrow \theta = \arctan\left(\frac{h+b}{x}\right) - \arctan\left(\frac{h}{x}\right)$$

(3) differentiate

$$\frac{d\theta}{dx} = \frac{1}{1 + \left(\frac{h+b}{x}\right)^2} \cdot \frac{-(h+b)}{x^2} - \frac{1}{1 + \left(\frac{h}{x}\right)^2} \cdot \frac{-h}{x^2}$$

$$= -\frac{x^2}{x^2 + (h+b)^2} \cdot \frac{(h+b)}{x^2} + \frac{x^2}{x^2 + h^2} \cdot \frac{h}{x^2}$$

$$= \frac{h}{x^2 + h^2} - \frac{h+b}{x^2 + (h+b)^2}$$

$$= \frac{h(x^2 + h^2 + 2hb + b^2) - (h+b)(x^2 + h^2)}{(x^2 + h^2)(x^2 + (h+b)^2)}$$

$$= \frac{b(h^2 + hb) - x^2}{(x^2 + h^2)(x^2 + (h+b)^2)}$$

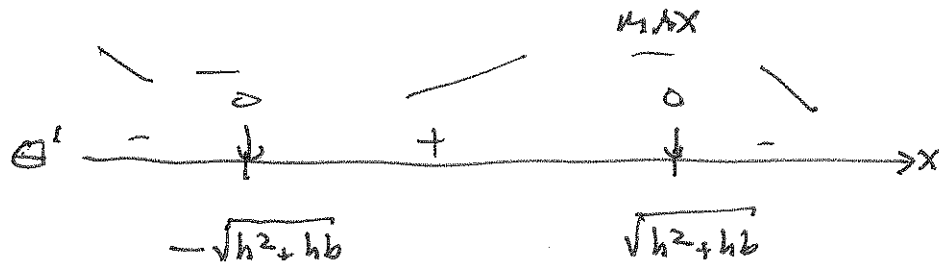
$$\frac{hx^2 + h^3 + 2h^2b + hb^2}{-hx^2 - h^3 - bh^2 - bx^2}$$

$$= \frac{h^2b + hb^2 - bx^2}{\text{which is } b(h^2 + hb - x^2)}$$

Note: the denominator  $> 0$   
and the numerator has  
zeros @  $x = \pm \sqrt{h^2 + hb}$



(4) sign diagram



(5) conclusion

The viewing angle is optimized when the observer is a distance of  $\sqrt{h^2+hb}$  from the building.

NOTE: This can also be done using the law of cosines... but this approach is very painful.