4.5: More Curve Sketching

ex1: $f(x) = x^4 - 4x^3 + 10$

ex2: $g(x) = x^3 - 27x$

ex3: $h(x) = \dfrac{(x+1)^2}{1+x^2}$

ex4: $f(x) = x^{2/3}(x^2 - 2x - 6)$

ex5: $g(x) = e^{2/x}$

ex4:

$$f(x) = x^{2/3}(x^2-2x-6)$$
$$= x^{2/3}(x-(1+\sqrt{7}))(x-(1-\sqrt{7}))$$
$$= x^{8/3} - 2x^{5/3} - 6x^{2/3}$$

solve $0 = x^2 - 2x - 6$

$$\Rightarrow x = \frac{2 \pm \sqrt{4-4(1)(-6)}}{2}$$
$$\Rightarrow x = \frac{2 \pm \sqrt{28}}{2}$$
$$\Rightarrow x = 1 \pm \sqrt{7}$$

$$f'(x) = \frac{8}{3}x^{5/3} - \frac{10}{3}x^{2/3} - \frac{12}{3}x^{-1/3}$$
$$= \frac{2}{3}x^{-1/3}(4x^{6/3} - 5x^{3/3} - 6)$$
$$= \frac{2}{3}x^{-1/3}(4x+3)(x-2)$$

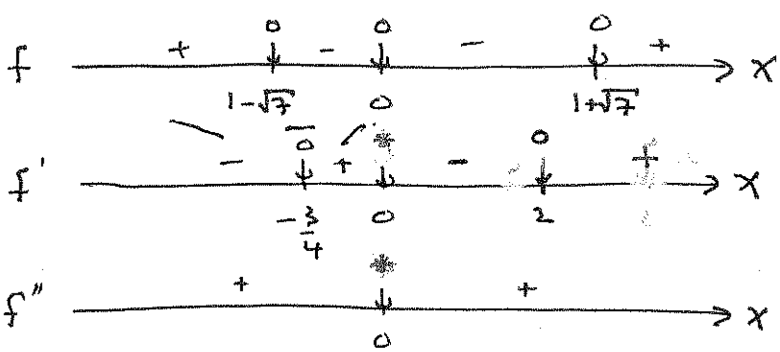
solve $0 = 4x^2 - 5x - 6$
$$= 4x^2 - 8x + 3x - 6$$
$$= 4x(x-2) + 3(x-2)$$
$$= (4x+3)(x-2)$$

$$f''(x) = \frac{40}{9}x^{2/3} - \frac{20}{9}x^{-1/3} + \frac{12}{9}x^{-4/3}$$
$$= \frac{4}{9}x^{-4/3}(10x^{+6/3} - 5x^{3/3} + 3)$$

solve $0 = 10x^2 - 5x + 3$

$$\Rightarrow x = \frac{5 \pm \sqrt{25 - 4(10)(3)}}{2(10)}$$

$\Rightarrow$ No real sol.



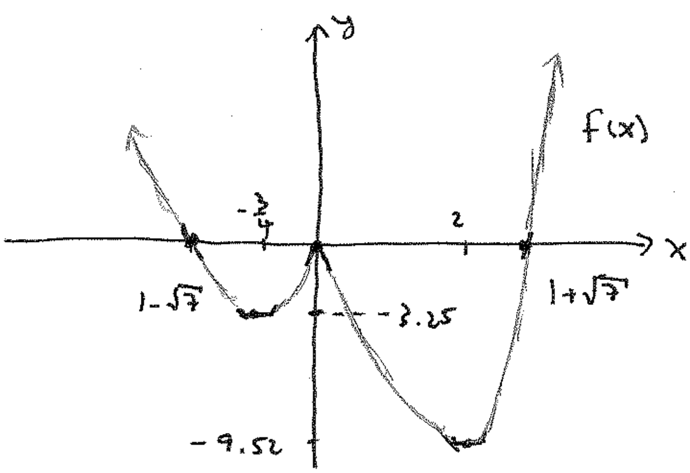
$f(1-\sqrt{7})$

$f(0)$

$f(1+\sqrt{7})$

$f(-\frac{3}{4}) \approx -3.25$

$f(2) \approx -9.52$



Guidelines for Sketching a Curve

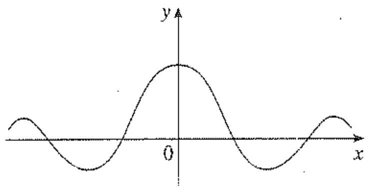
The following checklist is intended as a guide to sketching a curve $y = f(x)$ by hand. Not every item is relevant to every function. (For instance, a given curve might not have an asymptote or possess symmetry.) But the guidelines provide all the information you need to make a sketch that displays the most important aspects of the function.

A. Domain It's often useful to start by determining the domain D of f, that is, the set of values of x for which $f(x)$ is defined.

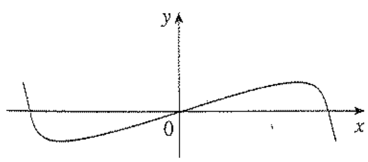
B. Intercepts The y-intercept is $f(0)$ and this tells us where the curve intersects the y-axis. To find the x-intercepts, we set $y = 0$ and solve for x. (You can omit this step if the equation is difficult to solve.)

C. Symmetry

(i) If $f(-x) = f(x)$ for all x in D, that is, the equation of the curve is unchanged when x is replaced by $-x$, then f is an **even function** and the curve is symmetric about the y-axis. This means that our work is cut in half. If we know what the curve looks like for $x \geqslant 0$, then we need only reflect about the y-axis to obtain the complete curve [see Figure 3(a)]. Here are some examples: $y = x^2$, $y = x^4$, $y = |x|$, and $y = \cos x$.



(a) Even function: reflectional symmetry



(b) Odd function: rotational symmetry

FIGURE 3

(ii) If $f(-x) = -f(x)$ for all x in D, then f is an **odd function** and the curve is symmetric about the origin. Again we can obtain the complete curve if we know what it looks like for $x \geqslant 0$. [Rotate 180° about the origin; see Figure 3(b).] Some simple examples of odd functions are $y = x$, $y = x^3$, $y = x^5$, and $y = \sin x$.

(iii) If $f(x + p) = f(x)$ for all x in D, where p is a positive constant, then f is called a **periodic function** and the smallest such number p is called the **period.** For instance, $y = \sin x$ has period 2π and $y = \tan x$ has period π. If we know what the graph looks like in an interval of length p, then we can use translation to sketch the entire graph (see Figure 4).

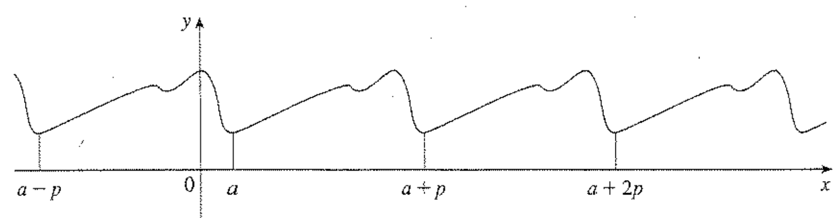


FIGURE 4
Periodic function: translational symmetry

D. Asymptotes

(i) *Horizontal Asymptotes.* Recall from Section 2.6 that if either $\lim_{x\to\infty} f(x) = L$ or $\lim_{x\to-\infty} f(x) = L$, then the line $y = L$ is a horizontal asymptote of the curve $y = f(x)$. If it turns out that $\lim_{x\to\infty} f(x) = \infty$ (or $-\infty$), then we do not have an asymptote to the right, but that is still useful information for sketching the curve.

(ii) *Vertical Asymptotes.* Recall from Section 2.2 that the line $x = a$ is a vertical asymptote if at least one of the following statements is true:

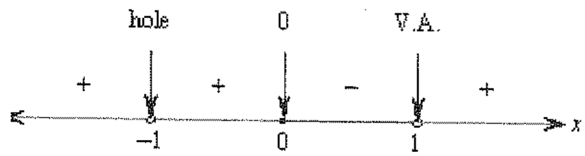
$$\boxed{1} \qquad \lim_{x\to a^+} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = \infty$$

$$\lim_{x\to a^+} f(x) = -\infty \qquad \lim_{x\to a^-} f(x) = -\infty$$

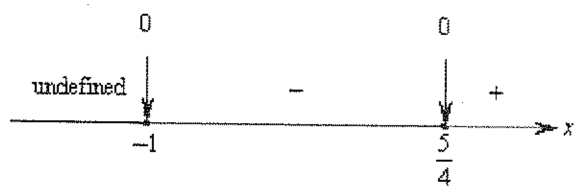
Examples:

What follows are two completed sign diagrams.

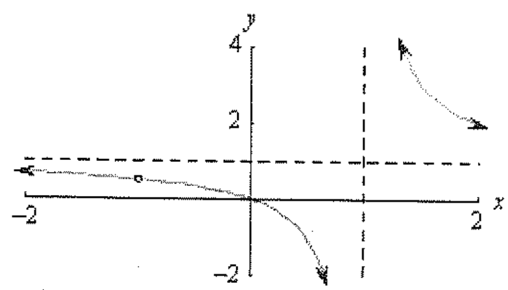
Example 1:



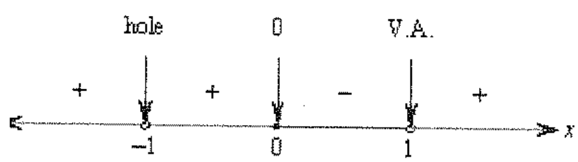
Example 2:



Example 3: Graph of $f\left(x\right)=\dfrac{x\left(x+1\right)}{\left(x-1\right)\left(x+1\right)}$



Sign Diagram of $f\left(x\right)$



The sign diagram seems to treat the vertical asymptote the same way as it does the hole.

Notice that the sign diagram does not contain information about the horizontal asymptote.