begin w/ claim C ... come back to A & B when they are needed.

3.3
1/4

Note: $\frac{d}{dx}\cos x = -\sin x$ in HW 3.3.20

Derive the other basic trig. derivatives.

Proving that $\frac{d}{dx}\sin(x) = \cos(x)$.

To prove that $\frac{d}{dx}\sin(x) = \cos(x)$, we first prove that $\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta} = 1$ and $\lim_{\theta\to 0}\frac{\cos(\theta)-1}{\theta} = 0$.

Claim A: $\lim_{\theta\to 0}\frac{\sin(\theta)}{\theta} = 1$.

To prove Claim A, we will use i.) trigonemetric geometry, ii.) the squeeze theorem, and iii.) we will call upon the symmetry of $\frac{\sin(\theta)}{\theta}$.

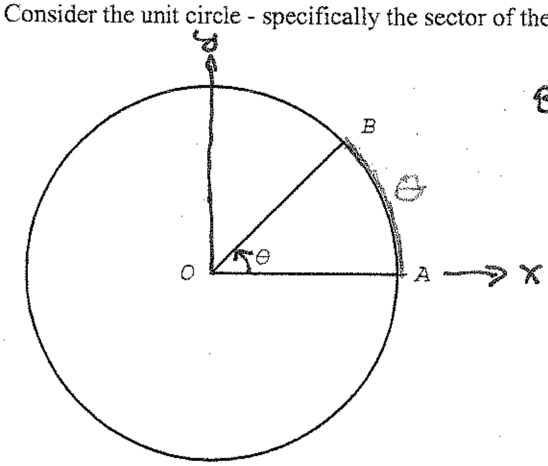
- **i.) Trigonometric Geometry**

prove using the squeeze thm so we need an upper & lower bound for $\frac{\sin\theta}{\theta}$ near $\theta = 0$.

- **Finding an upper bound to $\frac{\sin(\theta)}{\theta}$.**

Consider the unit circle - specifically the sector of the circle with center O, central angle $0 < \theta < \frac{\pi}{2}$, and radius 1.

assume in Q1.



B $(\cos\theta, \sin\theta)$

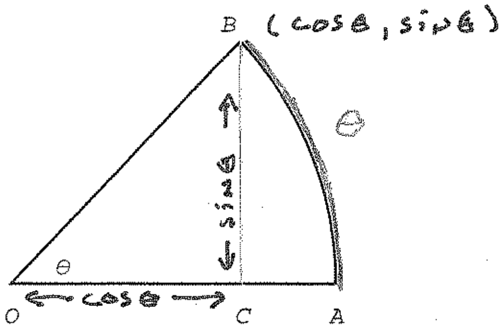
What are the coordinates of the point B:

How long is Arclength(AB):

Arclength(AB) = θ

(def of the radian)

Zooming in on the sector of the circle:



What is length(BC):

And since $\underbrace{\text{length(BC)}}_{\sin\theta} < \underbrace{\text{arclength(AB)}}_{\theta}$, we have that $\sin(\theta) < \theta$ and hence $\frac{\sin(\theta)}{\theta} < 1$.

- **Finding a lower bound to $\frac{\sin(\theta)}{\theta}$.**

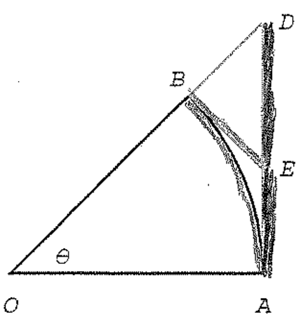
Let the tangents at A and B intersect at point E.

Now, consider the following lengths in relation to eachother and to θ.

length(EB) < length(ED) ← hypotenus

length(ED) > length(AE) since length(AE) = length(EB)

length(AD) = length(AE) + length(ED) = tan θ



we go thru this geometry argument to prove θ < tan θ

θ = arclength(AB) < length(AE) + length(EB). < $\underline{\text{length(AD)}}$ = tan θ

$\theta < \tan(\theta) \rightarrow \theta < \frac{\sin(\theta)}{\cos(\theta)} \rightarrow \cos(\theta) < \frac{\sin(\theta)}{\theta}$.

- **ii.) The Squeeze Theorem** we only consider $\theta > 0$ since we are in Q1.

In part i.) we showed that $\cos(\theta) < \frac{\sin(\theta)}{\theta} < 1$, for $0 < \theta < \frac{\pi}{2}$. Since $\lim_{\theta \to 0} \cos(\theta) = 1$and $\lim_{\theta \to 0} 1 = 1$, by the squeeze theorem we have that $\lim_{\theta \to 0^+} \frac{\sin(\theta)}{\theta} = 1$.

- **iii.) Symmetry Argument.** we use symmetry to address the case where $\theta < 0$.

If $f(\theta) = \frac{\sin(\theta)}{\theta}$, we have that $f(-\theta) = \frac{\sin(-\theta)}{-\theta} = \frac{-\sin(\theta)}{-\theta} = \frac{\sin(\theta)}{\theta} = f(\theta)$. Hence, $f(\theta) = \frac{\sin(\theta)}{\theta}$is an even function. This means the left and right limits must be equal and so $\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1$.

result: $\lim_{\theta \to 0} \frac{\sin\theta}{\theta} = 1$

Claim B: $\lim_{\theta \to 0} \frac{\cos(\theta) - 1}{\theta} = 0$.

To find, $\lim_{\theta \to 0} \frac{\cos(\theta) - 1}{\theta}$, multiply the expression by the "conjugate" of the numerator.

$$\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = \lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} \cdot \frac{\cos\theta + 1}{\cos\theta + 1}$$

$$= \lim_{\theta \to 0} \frac{\cos^2\theta - 1}{\theta \cdot (\cos\theta + 1)}$$

$$= \lim_{\theta \to 0} \frac{-\sin^2\theta}{\theta(\cos\theta + 1)}$$

$$= \lim_{\theta \to 0} \frac{\sin\theta}{\theta} \cdot \lim_{\theta \to 0} \frac{-\sin\theta}{\cos\theta + 1}$$

$$= 1 \cdot 0$$

$$= 0$$

result: $\lim_{\theta \to 0} \frac{\cos\theta - 1}{\theta} = 0$

Claim C: $\frac{d}{dx} \sin(x) = \cos(x)$.

Recall that $\sin(a + b) = \sin(a)\cos(b) + \cos(a)\sin(b)$.

$$\frac{d}{dx}\sin x = \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h}$$

$$= \lim_{h\to 0}\frac{\sin x\cos(h)+\cos x\sin(h)-\sin x}{h}$$

$$= \lim_{h\to 0}\left[\frac{\sin x(\cos(h)-1)}{h}+\frac{\cos x\sin(h)}{h}\right]$$

$$= \sin x\lim_{h\to 0}\frac{\cos h-1}{h}+\cos x\lim_{h\to 0}\frac{\sin(h)}{h}$$

$$= \sin x\cdot 0+\cos x\cdot 1$$

$$= \cos x$$