2.4: The Precise Definition of the Limit.

The calculus is built upon the limit concept. For 200 years, mathematicians relied on an intuitive understanding of limit... they trusted their feelings.

But there was a crisis in math in the 1800's that forced mathematicians to be more rigorous than "arbitrarily" & "sufficiently close."

Thanks to Cauchy & Weierstrass, we have the following definition:

Dfn: Let f be a fct defined on some open interval that contains the number a, except possibly @ a itself. Then we say that the limit of $f(x)$ as x approaches a is L and we write $\lim_{x \to a} f(x) = L$ if $\forall \varepsilon > 0 \ \exists \delta > 0$ s.t.

if $0 < |x - a| < \delta$ then $|f(x) - L| < \varepsilon$.

Explain the ε-δ trade by using the table & asking for ε and giving δ.
⋮
we need a relationship between ε & δ.

Generally, an ε-δ proof has two parts.

(1) Guessing a value for δ.

(2) Proof where you show δ works.

<u>Ex 1:</u> Prove $\lim_{x \to 2} (3x - 7) = -1$

□ <u>proof.</u>

Let $\varepsilon > 0$ be given.

choose $\delta = \frac{\varepsilon}{3}$

If $|x - 2| < \delta$

$\Rightarrow |x - 2| < \frac{\varepsilon}{3}$

$\Rightarrow 3|x - 2| < \varepsilon$

$\Rightarrow |3x - 6| < \varepsilon$

$\Rightarrow |(3x - 7) - (-1)| < \varepsilon$ ∎

Guessing δ.

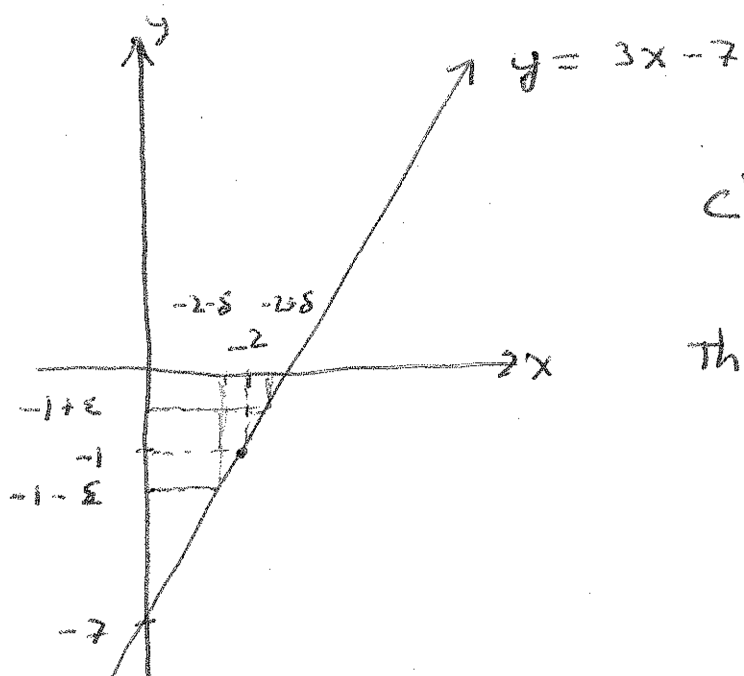
If $\varepsilon > |(3x - 7) - (-1)|$

$= |3x - 6|$

$= 3|x - 2|$

$\Rightarrow \frac{\varepsilon}{3} > |x - 2|$

so choose $\delta = \frac{\varepsilon}{3}$.



Choose ε

This dictates δ.

Ex2: Prove $\lim_{x \to 2} (x^2 + x - 1) = 5$.

□ proof.

Let $\varepsilon > 0$ be given.

choose $\delta = \min(1, \frac{\varepsilon}{6})$

If $0 < |x-2| < \delta$

$\Rightarrow |x-2| < \min(1, \frac{\varepsilon}{6})$

This means $|x-2| < 1 \Rightarrow 1 < x < 3$

$\Rightarrow 4 < x+3 < 6 \Rightarrow |x+3| < 6$

we also know $|x-2| < \frac{\varepsilon}{6}$

$\Rightarrow |x+3||x-2| < 6 \cdot \frac{\varepsilon}{6}$

$\Rightarrow |(x^2+x-1) - 5| < \varepsilon$ ∎

Guessing δ.

If $\varepsilon > |(x^2+x-1) - 5|$

$= |x^2+x-6|$

$= |(x+3)(x-2)|$

$= |x+3||x-2|$

since we care about x values near 2...

suppose $|x-2| < 1$

$\Rightarrow -1 < x-2 < 1$

$\Rightarrow 1 < x < 3$

and $4 < x+3 < 6$

$\Rightarrow |x+3| < 6$ when $1 < x < 3$

$\Rightarrow \varepsilon > 6|x-2|$

$\Rightarrow \frac{\varepsilon}{6} > |x-2|$

The Difference Law: If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$, then $\lim_{x\to a}(f(x) - g(x)) = L - M$

□ proof.

Let $\varepsilon > 0$ be given.

$\Rightarrow \exists\ \delta_1, \delta_2 > 0$ s.t. $0 < |x-a| < \delta_1 \Rightarrow |f(x) - L| < \frac{\varepsilon}{2}$

and $0 < |x-a| < \delta_2 \Rightarrow |g(x) - M| < \frac{\varepsilon}{2}$

Let $\delta = \min(\delta_1, \delta_2)$

$\Rightarrow$ If $0 < |x-a| < \delta$ then $|f(x)-L| + |g(x) - M| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$

and $|f(x)-L| + |M - g(x)| < \varepsilon$

which by the triangle inequality means

$$|(f(x)-L) + (M-g(x))| < \varepsilon$$

and $|(f(x) - g(x)) - (L-M)| < \varepsilon$ ■

Dfn. of an infinite limit: Let f be a fct defined on some open interval that contains a (except possibly a itself). Then $\lim_{x\to a} f(x) = \infty$ if $\forall\ M > 0\ \exists\ \delta > 0$ s.t. if $0 < |x-a| < \delta$ then $f(x) > M$.

ex3: Prove $\lim_{x\to 5} \frac{1}{(x-5)^4} = \infty$.