■ Finding the minimum distance between a function and a point

Overview
In class, an interesting question surfaced. Are various methods for finding the minimum distance between a point and a function equivalent? We know that they should be, and yet they seem different. So, beginning with a specific example, I attempted to answer this question. In the very first case, I solved the problem exactly and I solved it fully. However, in subsequent cases I reduced the problem as follows. Finding the minimum distance amounts to finding the best route. Finding the best route amounts to finding a specific *x* value. So, in later cases, I simply found the *x* value, or to be precise, I found the equation whose solution is that *x* value. At this point the solution is Q.E.D.

■ Find the minimum distance between the point $(-3, 1)$ and $y = x^2$.

■ a.) Using geometry.

We know from geometry that the minimum distance between a line *L* and point not on the line is along the line perpendicular to *L* that goes through the point.

In our present situation, we need the line perpendicular to $y = x^2$ that goes through the point $(-3, 1)$. We know that this curve has slopes $m = 2x$.

We want the line with slope $m = \frac{1}{-2x}$ that goes through the point $(-3, 1)$, so we have that $y - 1 = \frac{1}{-2x}(x - (-3))$. That is, $y = -\frac{3}{2x} + \frac{1}{2}$. But . . . this isn't a line. Why, we used a slope that is dependent upon x. In order to find the line, we need to find an x - specifically where $x^2 = -\frac{3}{2x} + \frac{1}{2}$. Clearing denominators, we have that $2x^3 = -3 + x$ or that $2x^3 - x + 3 = 0$.

This is a cubic function without a rational root (which can be seen by using the rational root theorem). So, we must resort to other methods to solve it, namely a computer algebra system like *Mathematica*. *Mathematica* gives three solutions, only one of which is real. The real solution is: $x = -\frac{6^{1/3} + (27 - \sqrt{723})^{2/3}}{6^{2/3}(27 - \sqrt{723})^{1/3}}$ which is approximately $x \cong -1.28962$.

For this value of x, $y = \left(-\frac{6^{1/3} + (27 - \sqrt{723})^{2/3}}{6^{2/3}(27 - \sqrt{723})^{1/3}}\right)^2$ which is approximately $y \cong 1.66313$.

The distance between $(-3, 1)$ and $\left(-\frac{6^{1/3} + (27 - \sqrt{723})^{2/3}}{6^{2/3}(27 - \sqrt{723})^{1/3}}, \left(-\frac{6^{1/3} + (27 - \sqrt{723})^{2/3}}{6^{2/3}(27 - \sqrt{723})^{1/3}}\right)^2\right)$ is distance $= \sqrt{\left(3 + \frac{1}{(162 - 6\sqrt{723})^{1/3}} + \frac{1}{6}\left(162 - 6\sqrt{723}\right)^{1/3}\right)^2 + \left(1 - \frac{\left(6^{1/3} + (27 - \sqrt{723})^{2/3}\right)^2}{6\,6^{1/3}(27 - \sqrt{723})^{2/3}}\right)^2}$ which is approximately distance $\cong 4.34058$.

■ b.) Using the distance formula

The distance between the point $(-3, 1)$ and the curve $y = x^2$ is $d(x) = \sqrt{(x - (-3))^2 + (x^2 - 1)^2}$. This has a minimum where $d'(x) = 0$ and $d''(x) > 0$.

$d'(x) = \frac{2(x+3)+4x(x^2-1)}{2\sqrt{\sqrt{(x+3)^2+(x^2-1)^2}}} = \frac{2x^3-x+3}{\sqrt{(x+3)^2+(x^2-1)^2}}$. Since the denominator is always positive, this has zeroes where $2x^3 - x + 3 = 0$. This is the same cubic we solved above. However, in order to see that the zero (found above) is a minimum, let's find the second derivative.

After some simplification, we find $d''(x) = \frac{-19+x^2(60+x(24-3x+2x^3))}{(10+x(6-x+x^3))^{3/2}}$. Evaluating this at zero (found above), we have:

$$d''\left(-\frac{6^{1/3}+\left(27-\sqrt{723}\right)^{2/3}}{6^{2/3}\left(27-\sqrt{723}\right)^{1/3}}\right) =$$

$$-\left(12\left(1110\left(4347\sqrt{3}-485\sqrt{241}\right)\left(27-\sqrt{723}\right)^{1/3}+6^{1/3}\left(323551\sqrt{3}-36099\sqrt{241}\right)\left(27-\sqrt{723}\right)^{2/3}+\right.\right.$$
$$\left.\left.3\,2^{2/3}\,3^{1/6}\left(-13468879+500913\sqrt{723}\right)\right)\right)\Big/$$
$$\left(\left(27-\sqrt{723}\right)^{11/6}\sqrt{\frac{485\,6^{2/3}-54\,2^{2/3}\,3^{1/6}\sqrt{241}+6^{1/3}\left(244-9\sqrt{723}\right)\left(27-\sqrt{723}\right)^{2/3}-118\left(27-\sqrt{723}\right)^{4/3}}{-27+\sqrt{723}}}\right.$$
$$\left.\left(-485\,6^{2/3}+54\,2^{2/3}\,3^{1/6}\sqrt{241}+118\left(27-\sqrt{723}\right)^{4/3}+6^{1/3}\left(27-\sqrt{723}\right)^{2/3}\left(-244+9\sqrt{723}\right)\right)\right)$$

This may be the nastiest expression we have ever seen, but what matters is that it is approximately 4.89459 which is positive. So, we have again found the minimum distance.

▪ c.) Using the square of the distance.

Lemma.
Suppose $[f(x)]^2$ with $f(x) > 0$ is a twice differentiable function with a local minimum at $x = a$. Then $f(x)$ also has a minimum at $x = a$.

□ Proof.
$\frac{d}{dx}[f(x)]^2 = 2f(x)f'(x)$. Evaluating at $x = a$ gives a zero since there is a minimum on a twice differentiable function, but $f(a) > 0$, so $f'(a) = 0$.
$\frac{d^2}{dx^2}[f(x)]^2 = 2[f'(x)]^2 + 2f(x)f''(x)$. Evaluating at $x = a$ gives a positive result since there is a minimum, but we already showed that $f'(a) = 0$ and we assumed $f(a) > 0$, se we have that $f''(a) > 0$.
Hence, $f(x)$ has a minimum at $x = a$. ■

Consider the function $D(x) = [d(x)]^2 = (x+3)^2 + (x^2-1)^2$.
$D'(x) = 2(x+3) + 4x(x^2-1)$. This looks precisely like the numerator we found in $d'(x)$.
It has a real zero when $4x^3 - 2x + 6 = 0$ or $2x^3 - x + 3 = 0$. We know this is a minimum because $D''(x) = 12x^2 - 2$ is positive at the critical number of $D'(x)$.

Find the minimum between a function and a point.

a.) Using geometry.

Find the minimum between a smooth function $y = f(x)$ and the point (a, b). Using geometry we have that the slopes on $f'(x)$ are given by $-\frac{1}{f'(x)}$. Thus the lines perpendicular to $f(x)$ are given by $y - b = -\frac{1}{f'(x)}(x - a)$. That is, $y = -\frac{1}{f'(x)}(x - a) + b$. Furthermore, we know that $y = f(x)$ and so we can fix the slope of this line by solving $f(x) = -\frac{1}{f'(x)}(x - a) + b$. Moving everything to the left side and clearing the denominators (we assume $f'(x) \neq 0$), we have that $(x - a) + f'(x)(f(x) - b) = 0$.

b.) Using the square of the distance.

Let $D(x) = (x - a)^2 + (f(x) - b)^2$. Then, $D'(x) = 2(x - a) + 2 f'(x)(f(x) - b)$. This has critical numbers when $2(x - a) + 2 f'(x)(f(x) - b) = 0$. Dividing by two, we have that $(x - a) + f'(x)(f(x) - b) = 0$ which is the same result as above. Thus, we have shown that the two methods are equivalent.