13.3: Arclength and Curvature

Recall from 10.2 that arclength $L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$.

The arclength of the space curve $\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$

is $L = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}\, dt$

$= \int_a^b |\vec{r}\,'(t)|\, dt$

this is assuming the curve is traversed exactly once

on $a \le t \le b$.

<u>Ex1</u>: Find the length of $\vec{r}(t) = \langle 3\sin(t), 7t, -3\cos(t)\rangle$

on $-5 \le t \le 5$.

If C is a piecewise smooth curve given by

$\vec{r}(t) = \langle x(t), y(t), z(t)\rangle$ on $a \le t \le b$ and at least

one of the functions x, y, z is 1–1 on (a,b),

then the arclength function s is:

$$s(t) = \int_a^t |\vec{r}\,'(u)|\, du$$

Differentiating w/respect to t gives

$$\frac{ds}{dt} = s'(t) = |\vec{r}\,'(t)|$$

EX2: Reparametrize $\vec{r}(t) = \langle 3\sin(t), 7t, -3\cos(t) \rangle$ writ the arclength measured from $(0, 0, -3)$ in the direction of increasing t.

Soln:

$(0, 0, -3) \iff t = 0$.

recall from (ex1) that $|\vec{r}\,'(t)| = \sqrt{58} = \frac{ds}{dt}$

$\Rightarrow \quad s = \int_0^t \sqrt{58}\, du$

$= \sqrt{58}\, t$

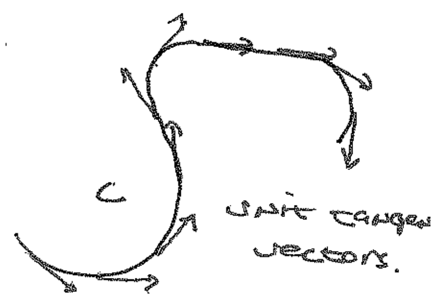
$\Rightarrow \quad t = \frac{s}{\sqrt{58}}$

So, $r[t(s)] = \left\langle 3\sin\left(\frac{s}{\sqrt{58}}\right), \frac{7s}{\sqrt{58}}, -3\cos\left(\frac{s}{\sqrt{58}}\right) \right\rangle$ //.

Recall that the unit tangent vector $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$.

Defn: The curvature k of a curve is

$$k = \left| \frac{d\vec{T}}{ds} \right| \quad \text{(Dfn \#1)}$$



This should make intuitive sense.

This 1st. defn. of k is extremely difficult. So notice we can use the chain rule

$$\frac{d\vec{T}}{ds} = \frac{d\vec{T}}{dt}\frac{dt}{ds} = \frac{\frac{d\vec{T}}{dt}}{\frac{ds}{dt}} = \frac{\vec{T}\,'(t)}{|r'(t)|}$$

and $$k = \frac{|\vec{T}\,'(t)|}{|r'(t)|} \quad \text{(Dfn \# 2)}$$

Our goal is to find an expression for the osculating circle ... so we need the relationship between the curvature & radius of a circle.

Ex 3: Show that the curvature of a circle w/radias a is $k = \frac{1}{a}$.

We want an even simpler way to calculate/work w/curvature... but we need 3 facts.

#1 $\vec{V} \times \vec{V} = \vec{0}$

#2 $\frac{d}{dt}\left[f(t)\vec{u}(t)\right] = f'(t)\vec{u}(t) + f(t)\vec{u}'(t)$
(product rule)

#3 If $|\vec{r}(t)| = c$ (a constant), then $\vec{r}'(t) \perp \vec{r}(t)$ for all t.

□ proof.

$\vec{r}(t) \cdot \vec{r}(t) = |\vec{r}(t)|^2 = c^2$

$\Rightarrow \vec{r}(t) \cdot \vec{r}(t) = c^2$

differentiate both sides using the product rule

$\Rightarrow \vec{r}'(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{r}'(t) = 0$

$\Rightarrow 2\vec{r}'(t) \cdot \vec{r}(t) = 0$

$\Rightarrow \vec{r}'(t) \cdot \vec{r}(t) = 0$

Hence $\vec{r}(t)$ & $\vec{r}'(t)$ are always orthogonal.

#4 $\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$

Now we are ready to derive our formula

$$\vec{T} = \frac{\vec{r}\,'}{|\vec{r}\,'|} \Rightarrow \vec{r}\,' = |\vec{r}\,'|\,\vec{T}$$

$$\Rightarrow \vec{r}\,' = \frac{ds}{dt}\vec{T}$$

Differentiate both sides using the product rule on R.H.S.

$$\Rightarrow \vec{r}\,'' = \frac{d^2s}{dt^2}\vec{T} + \frac{ds}{dt}\vec{T}\,'$$

$$\Rightarrow \vec{r}\,' \times \vec{r}\,'' = \frac{ds}{dt}\vec{T} \times \left(\frac{d^2s}{dt^2}\vec{T} + \frac{ds}{dt}\vec{T}\,'\right)$$

$$= \frac{ds}{dt}\frac{d^2s}{dt^2}(\vec{T}\times\vec{T}) + \left(\frac{ds}{dt}\right)^2(\vec{T}\times\vec{T}\,')$$

$$= \left(\frac{ds}{dt}\right)^2(\vec{T}\times\vec{T}\,')$$

$$\Rightarrow |\vec{r}\,' \times \vec{r}\,''| = \left(\frac{ds}{dt}\right)^2 |\vec{T}\times\vec{T}\,'|$$

since $\vec{T} \perp \vec{T}\,'$

$$= \left(\frac{ds}{dt}\right)^2 |\vec{T}||\vec{T}\,'|$$

$$= \left(\frac{ds}{dt}\right)^2 |\vec{T}\,'|$$

Thus $|\vec{T}\,'| = \dfrac{|\vec{r}\,' \times \vec{r}\,''|}{\left(\frac{ds}{dt}\right)^2}$

$$= \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^2}$$

> If you have a plane curve $y = f(x)$
>
> $k(x) = \dfrac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$ (Dfn #4)

recall $k = \dfrac{|\vec{T}\,'|}{|\vec{r}\,'|}$

so $k = \dfrac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3}$ (Dfn #3)

Ex 4: Consider $r(t) = \langle \cos t, \sin t, \ln(\cos t) \rangle$

Find the curvature.

Normal vector: since $|\vec{T}(t)| = 1$, $\vec{T} \cdot \vec{T}' = 0$

So we define the unit normal vector

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$$

and we define the binormal vector $\vec{B}(t) = \vec{T}(t) \times \vec{N}(t)$

<u>Ex4 rev.</u>: (a) Find $\vec{T}$, $\vec{N}$, $\vec{B}$

(b) Find $\vec{T}, \vec{N}, \vec{B}$ @ $(1,0,0)$

(c) eqt of the ~~tangent~~ osculating plane.

(d) eqt of the kissing circle...

$\vec{r}(t) = \langle \cos t, \sin t, \ln(\cos t) \rangle$ $\qquad t = 0 + 2k\pi,\ k \in \mathbb{Z}$

$\vec{r}'(t) = \langle -\sin t, \cos t, -\tan t \rangle$

$$|\vec{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t + \tan^2 t}$$

$$= \sqrt{1 + \tan^2 t}$$

$$= |\sec t|$$

Find κ after evaluating $\vec{r}'(0)$ and $\vec{r}''(0)$.

$\vec{r}''(t) = \langle -\cos t, -\sin t, -\sec^2 t \rangle$

$$\vec{r}'(t) \times \vec{r}''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -s & c & -t \\ -c & -s & -\sec^2 t \end{vmatrix}$$

$$= \left\langle -\sec t - \frac{\sin^2 t}{\cos t}, -\frac{\sin t}{\cos^2 t} + \sin t, 1 \right\rangle$$

$$= \left\langle -\frac{1 + \sin^2 t}{\cos t}, \sin t(1 - \sec^2 t), 1 \right\rangle \Big|_{t=0} \langle -1, 0, 1 \rangle$$

$$|\vec{r}'(t) \times \vec{r}''(t)| \Big|_{t=0} = \sqrt{2}$$

$$\hookrightarrow \quad k = \left.\frac{|r'(t) \times r''(t)|}{|r'(t)|^3}\right|_{t=0} \quad \frac{\sqrt{2}}{1} = \sqrt{2}$$

$$\vec{T}(t) = \frac{1}{|\sec t|} \langle -\sin t \,,\ \cos t \,,\ -\tan t \rangle$$

$$= \left.\left\langle -\tan t \,,\ 1 \,,\ \frac{-\sin t}{\cos^2 t} \right\rangle\right|_{t=0} \quad \langle 0, 1, 0 \rangle$$

$$\vec{T}'(t) = \left\langle -\sec^2 t \,,\ 0 \,,\ \frac{-\cos t \cdot \cos^2 t - 2\cos t \cdot (-\sin t)(-\sin t)}{\cos^4 t} \right\rangle$$

$$= \left.\left\langle -\sec^2 t \,,\ 0,\ -\sec t - \frac{2\sin^2 t}{\cos^3 t} \right\rangle\right|_{t=0} \quad \langle -1, 0, -1 \rangle$$

$$|\vec{T}'(0)| = \sqrt{2}$$

$$\vec{N}(t) = \left\langle \frac{-1}{\sqrt{2}} , 0 , \frac{-1}{\sqrt{2}} \right\rangle$$

$$\vec{B}(t) = \begin{vmatrix} \vec{i} & \vec{j} & k \\ 0 & 1 & 0 \\ -\frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{vmatrix} = \left\langle -\frac{1}{\sqrt{2}} , 0 \ \frac{1}{\sqrt{2}} \right\rangle$$

osculating plane: $-\frac{1}{\sqrt{2}}(x-1) + 0(y-0) + \frac{1}{\sqrt{2}}(z-0) = 0$

kissing circle. w/Radius $R = \frac{1}{k}$

$$\frac{1}{k}\vec{T}\cos\theta + \frac{1}{k}\vec{N}\sin\theta + \vec{r}(t) + \frac{1}{k}\vec{N}(t)$$

$$\Rightarrow \vec{r}(\theta) = \frac{1}{\sqrt{2}}\langle 0, 1, 0 \rangle \cos\theta + \frac{1}{\sqrt{2}}\left\langle -\frac{1}{\sqrt{2}} , 0, -\frac{1}{\sqrt{2}} \right\rangle \sin\theta$$

$$+ \langle 1, 0, 0 \rangle + \frac{1}{\sqrt{2}}\left\langle -\frac{1}{\sqrt{2}} , 0, -\frac{1}{\sqrt{2}} \right\rangle$$