

12.4
1/4

## 12.4: Cross Product

### Determinants

Ex 1: Is there a unique solution to the system

$$\begin{cases} x + 2y = c_1 \\ 3x + 4y = c_2 \end{cases}$$

To answer this, calculate the determinate

$$\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1(4) - 2(3).$$

If the determinate is not zero, then a unique solution exists.

Ex 2: Is there a unique solution to the system

$$\begin{cases} x + 2y + 3z = c_1 \\ 4x + 5y + 6z = c_2 \\ 7x + 8y + 9z = c_3 \end{cases}$$

calculate the determinate

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$

$$= 1(45 - 48) - 2(36 - 42) + 3(32 - 35)$$

$$= -3 + 12 - 9$$

$$= 0$$

So, there is not a unique solution.

12.4
2/4

Definition If  $\vec{u} = \langle u_1, u_2, u_3 \rangle$  and  $\vec{v} = \langle v_1, v_2, v_3 \rangle$ ,  
then the cross product is the vector

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \langle u_2 v_3 - u_3 v_2, \dots \rangle$$

Ex 3: If  $\vec{u} = \langle 1, 2, 3 \rangle$  and  $\vec{v} = \langle 4, 5, 6 \rangle$ ,  
find  $\vec{u} \times \vec{v}$ .

Ex 4: If  $\vec{u} = \langle 1, -1, 2 \rangle$  and  $\vec{v} = \langle -3, 5, -8 \rangle$ , find  
 $\vec{u} \times \vec{v}$ .

Ex 5: Show  $\vec{u} \times \vec{u} = \vec{0} \quad \forall \vec{u} \in \mathbb{V}_3$

Thm: The vector  $\vec{u} \times \vec{v}$  is orthogonal to  $\vec{u}$  and  $\vec{v}$ .

Ex 6: Show that  $\vec{u} \times \vec{v}$  in (ex 3) is orthogonal  
to  $\vec{u}$  and  $\vec{v}$  (dot product). Could also  
find the angle between  $\vec{u} \times \vec{v}$  &  $-\vec{u}$ .

Geometrically, there are two directions that could  
be taken by a vector  $\perp$  to the span of  
 $\vec{u}$  &  $\vec{v}$ .  $\vec{u} \times \vec{v}$  has direction fixed by the  
right-hand rule.

12.4
3/4

Now that we have the direction of  $\vec{u} \times \vec{v}$ , what is its magnitude?

Thm: If  $\theta$  is the angle between  $\vec{u}$  &  $\vec{v}$  ( $0 \leq \theta \leq \pi$ ), then  $|\vec{u} \times \vec{v}| = |\vec{u}| |\vec{v}| \sin \theta$

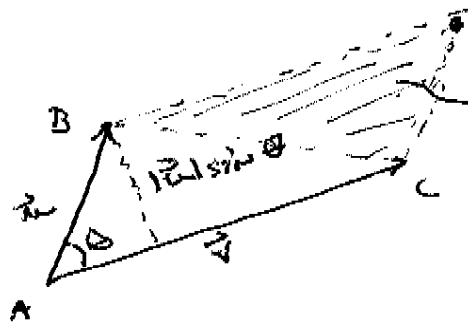
$\square$  proof.

In the book, ... read it backwards.

Corollary! Two non-zero vectors  $\vec{u}$  &  $\vec{v}$  are parallel iff  $\vec{u} \times \vec{v} = \vec{0}$ .

EX 7: Find a vector  $\perp$  to the plane thru  $A(1, 2, 3)$ ,  $B(4, 5, 6)$ , and  $C(7, 8, 9)$ .

EX 8: Find the area of the triangle w/ vertices  $A(1, 2, 3)$ ,  $B(4, 5, 6)$ , and  $C(7, 8, 9)$



Area of triangle is  $\frac{1}{2}$  the area of the parallelogram.  $= \frac{|\vec{u}| |\vec{v}| \sin \theta}{2}$

OR, the parallelogram has area  $|\vec{u} \times \vec{v}|$ .

That is, the length of  $\vec{u} \times \vec{v}$  equals the area of the parallelogram determined by  $\vec{u}$  and  $\vec{v}$ .

12.4
9/4

## Cross product properties

IF  $\vec{u}, \vec{v}, \vec{w} \in \mathbb{V}_3$  and  $c$  is a scalar

$$1) \vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$$

$$2) (c\vec{u}) \times \vec{v} = c(\vec{u} \times \vec{v}) = \vec{u} \times (c\vec{v})$$

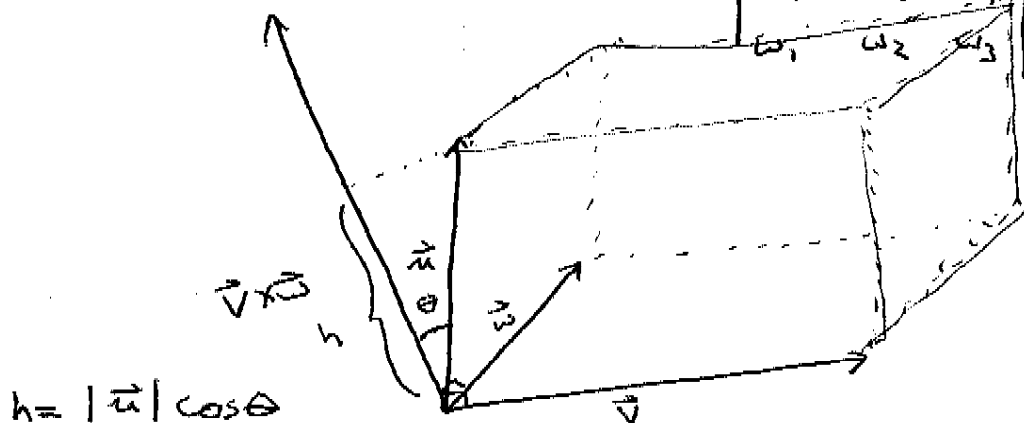
$$3) \vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$$

$$4) (\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$$

$$5) \vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w} \quad \text{Scalar triple product.}$$

$$6) \vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$$

Notice that  $\vec{u} \cdot (\vec{v} \times \vec{w}) =$



$$V = A \cdot h = |\vec{v} \times \vec{w}| |\vec{u}| \cos \theta$$

$$= |\vec{u}| |\vec{v} \times \vec{w}| \cos \theta$$

$$= |\vec{u} \cdot (\vec{v} \times \vec{w})| \quad \text{volume of the parallelepiped determined by } \vec{u}, \vec{v}, \text{ and } \vec{w}.$$

coplanar?