Abs. convergence & the ratio and root tests (+)

Notice that $\boxed{1}\ \sum \frac{(-1)^{n+1}}{n^2}$ and $\sum \frac{1}{n^2}$ both converge.

On the other hand.

$\boxed{2}\ \sum \frac{(-1)^{n+1}}{n}$ converges and $\sum \frac{1}{n}$ diverges.

The 1st is an example of an absolutely convergent series while the 2nd is conditionally convergent.

Defn. $\sum a_n$ is absolutely convergent if $\sum |a_n|$ converges.

Defn. $\sum a_n$ is conditionally convergent if it is convergent, but not absolutely convergent.

Ex1: Determine the convergence or divergence of $\sum_{n=1}^{\infty} \frac{\cos(\pi n)}{n^3}$.

Thm: If a series is abs. conv, then it is convergent. (proof in text).

11.6
2/6

The Ratio Test

i) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$, then $\sum a_n$ converges.

ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$, then $\sum a_n$ diverges.

iii) else it is inconclusive.

Ex 2: Test $\sum_{n=1}^{\infty} \frac{10^n}{(n+1)\,4^{2n+1}}$ for convergence.

consider

$$\lim_{n\to\infty}\left|\frac{\frac{10^{n+1}}{(n+2)\,4^{2(n+1)+1}}}{\frac{10^n}{(n+1)\,4^{2n+1}}}\right| = \lim_{n\to\infty}\left|\frac{10\cdot 10^n}{10^n}\cdot\frac{(n+1)\cdot 4^{2n+1}}{(n+2)\cdot 16\cdot 4^{2n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{10}{16}\cdot\frac{n+1)}{n+2)}\right| = \frac{10}{16}$$

Hence, the series converges by the ratio test.

□ proof of the ratio test.

(ii) If $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| > 1$, then $\exists N$ s.t. $n > N$ implies

$$\left|\frac{a_{n+1}}{a_n}\right| > 1 \Rightarrow |a_{n+1}| > |a_n| \text{ for } n > N.$$

but this implies $\lim_{n\to\infty} a_n \neq 0$ and so

$\sum a_n$ diverges.

11.6
3/6

(i) We are going to show convergence by creating a convergent geometric series that is above $\sum |a_n|$ and thus showing convergence by a comparison.

We have $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L < 1$. So, $\exists\, r$ s.t. $L < r < 1$. Since $L < r$, $\exists\, N$ s.t. $n \geq N$ implies $\left|\frac{a_{n+1}}{a_n}\right| < r$ for $n \geq N$.

$\Rightarrow |a_{n+1}| < |a_n| r$ for $n \geq N$.

$\Rightarrow |a_{N+1}| < |a_N| r$

and $|a_{N+2}| < |a_{N+1}| r < |a_N| r^2$

and $|a_{N+3}| < |a_{N+2}| r < |a_{N+1}| r^2 < |a_N| r^3$

$\vdots$

and in general, $|a_{N+k}| < |a_N| r^k$ for $k \geq 1$.

Now $\sum_{k=1}^{\infty} |a_N| r^k$ is a convergent geometric series.

and by comparison, $\sum_{n=N+1}^{\infty} |a_n|$ is convergent.

$\Rightarrow \sum_{n=1}^{\infty} |a_n|$ is convergent, so $\sum_{n=1}^{\infty} a_n$ is absolutely convergent. ∎

11.6
4/4

Ex 3: Test $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ for convergence.

Ratio Test.

$$\lim_{n\to\infty} \left| \frac{\frac{(n+1)!}{(n+1)^{(n+1)}}}{\frac{n!}{n^n}} \right| = \lim_{n\to\infty} \left| \frac{\cancel{(n+1)}\cancel{n!}\, n^n}{\cancel{n!}\,\cancel{(n+1)}(n+1)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{n^n}{(n+1)^n} \right|$$

$$= \lim_{n\to\infty} \left(\frac{n}{n+1} \right)^n$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n} \right)^{-n}$$

$$= \left(\lim_{n\to\infty} \left(1 + \frac{1}{n} \right)^n \right)^{-1}$$

$$= e^{-1} < 1.$$

so, the series is convergent absolutely.

Root Test

i) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$, $\sum a_n$ converges absolutely.

ii) If $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$, $\sum a_n$ diverges.

iii) else, the root test is inconclusive.

11.6
5/6

Ex4: Test $\sum_{n=1}^{\infty} \frac{1}{n}$ using the root test.

$$\lim_{n\to\infty} \sqrt[n]{\frac{1}{n}} = \lim_{n\to\infty} n^{-\frac{1}{n}}$$ indeterminate form.

$$= \lim_{x\to\infty} e^{-\frac{1}{x}\ln(x)}$$

$$= e^{\lim_{x\to\infty} \frac{-\ln(x)}{x}}$$

$$\overset{H}{=} e^{\lim_{x\to\infty} -\frac{1}{x}}$$

$$= e^{0}$$

$$= 1$$ inconclusive, but we know the harmonic series diverges

Ex5: Test $\sum \frac{1}{n^2}$ using the root test.

$$\lim_{n\to\infty} \sqrt[n]{n^{-2}} = \lim_{n\to\infty} n^{-2/n}$$ indeterminate form.

$$= e^{\lim_{n\to\infty} \frac{-2\ln(x)}{x}}$$

$$= e^{\lim_{n\to\infty} \frac{-2}{x}}$$

$$= e^{0}$$

$$= 1.$$ inconclusive, but we know that the series is convergent by the p-test.

(ii) claim: If $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$, then $\sum_1 a_n$ diverges.

□ proof.

If $\lim_{n\to\infty} \sqrt[n]{|a_n|} > 1$, then $\exists N$ s.t. $n > N$ implies $\sqrt[n]{|a_n|} > 1$. This in turn implies that $|a_n| > 1^n > 1$ for $n > N$. So, $\lim_{n\to\infty} a_n \neq 0$ and $\sum_1 a_n$ is divergent. ■

(i) claim: If $\lim_{n\to\infty} \sqrt[n]{a_n} = L < 1$, then $\sum_1 a_n$ converges.

□ proof.

As w/ the ratio test, we will show convergence by comparing to a convergent geometric series.

We have $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L < 1$. So, $\exists r$ s.t. $L < r < 1$. Since $L < r$, $\exists N$ s.t. $n \geq N$ implies $\sqrt[n]{|a_n|} < r < 1$ for $n \geq N$. But this means $|a_n| < r^n < 1$ for $n \geq N$. Summing the terms when $n \geq N$ we have.

$$\sum_{n=N}^{\infty} |a_n| < \sum_{n=N}^{\infty} r^n$$ which is a convergent geometric series.

Thus $\sum_{n=N}^{\infty} |a_n|$ converges ~~absolutely~~ by comparison and $\sum_{n=1}^{\infty} a_n$ converges absolutely. ■

Still need to cover rearrangements