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11.4: The Comparison Test

Does $\sum_{n=1}^{\infty} \frac{1}{3^n + 2}$ converge or diverge?

well, this reminds us of $\sum_{n=1}^{\infty} \frac{1}{3^n}$ which is a convergent geometric series.

The Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series of positive terms.

- If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n , then $\sum a_n$ is also convergent.
- If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n , then $\sum a_n$ is also divergent.

□ proof of (i).

$$\text{Let } s_n = \sum_{i=1}^n a_i, \quad t_n = \sum_{i=1}^n b_i, \quad \text{and } t = \sum_{i=1}^{\infty} b_i$$

Both sequences $\{a_n\}$ and $\{b_n\}$ are ~~positive~~ positive so the partial sums are increasing. since $t_n \rightarrow t$, so $t_n \leq t \forall n$. Since $a_i \leq b_i$, we have $s_n \leq t_n$. Thus, $s_n \leq t \forall n$. This means $\{s_n\}$ is increasing and bounded above and therefore converges by the monotonic sequence theorem. ■

The proof of (ii) is in the text.

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NOTE: To use the comparison test you must have a known series $\sum b_n$ to compare w/. Generally, we compare use a p-series or a geometric series.

EX 1: Does $\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3}$ converge or diverge?

EX 2: Does $\sum_{n=1}^{\infty} \frac{\sin^2 x + 1}{n}$ converge or diverge?

EX 3: Does $\sum_{n=1}^{\infty} \frac{1}{3^{n-2}}$ converge or diverge?

we do not know using the comparison test...

The Limit Comparison Test

Suppose that $\sum a_n$ and $\sum b_n$ are series w/ positive terms. If $\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c$ where c is a finite positive number. Then either both series converge or both diverge.

□ proof.

Let $m, M > 0$ be s.t. $m < c < M$. Because $\frac{a_n}{b_n}$ is close to c for large n , so $\exists N$

$$\text{s.t. } m < \frac{a_n}{b_n} < M, \quad n > N$$

$$\Rightarrow mb_n < a_n < Mb_n, \quad n > N$$

If $\sum b_n$ converges, so does $\sum Mb_n$ and $\sum a_n$ converges.

If $\sum b_n$ diverges, so do $\sum mb_n$ and $\sum a_n$ diverges.

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Ex 3 rev: Does $\sum_{n=1}^{\infty} \frac{1}{3^{n-2}}$ converge or diverge?

Ex 4: Does $\sum_{n=2}^{\infty} \frac{n^3+2}{n^4-9}$ converge or diverge?

Estimating Sums.

We can estimate the remainder by using our convergent $\sum b_n$ that is above $\sum a_n$. An upper-bound on the remainder of $\sum b_n$ is also an upper-bound on the remainder of $\sum_{i=1}^n a_i$.

Ex 5: Approximate $\sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$ to within 0.0005 using the ~~smallest~~ least possible terms in the partial sum.... I have an upperbound.

$$\underbrace{\sum_{n=1}^{\infty} \frac{1}{n^{5/2}}}_S \leq \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^5}}_T$$

for what n is $|T - T_n| \leq 0.0005$, by our Remainder Est. using the integral test.

$$S_n = \sum_{i=1}^n \frac{1}{i^{5/2}} \leq \sum_{i=1}^n \frac{1}{i^5} = T_n \quad \text{and}$$

$$\sum_{i=n+1}^{\infty} \frac{1}{i^{5/2}} \leq \sum_{i=n+1}^{\infty} \frac{1}{i^5} \leq \int_n^{\infty} \frac{dx}{x^5} \leq 0.0005.$$

solve for n .