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10.2: Calculus w/parametric curves

Our goal is to explore vector fct which are related to parametric curves... An overview of the calculus of parametric curves is as follows

If $x(t)$ and $y(t)$ are parametric functions for x and y, then...

$\boxed{1}$ $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}, \quad \frac{dx}{dt} \neq 0.$$

$\boxed{2}$ $$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}}, \quad \frac{dx}{dt} \neq 0.$$

Ex1: Sketch a graph of $x = 10 - t^2$ and $y = t^3 - 12t$

$x = (\sqrt{5}+t)(\sqrt{5}-t)$ and $y = t(t+2\sqrt{3})(t-2\sqrt{3})$

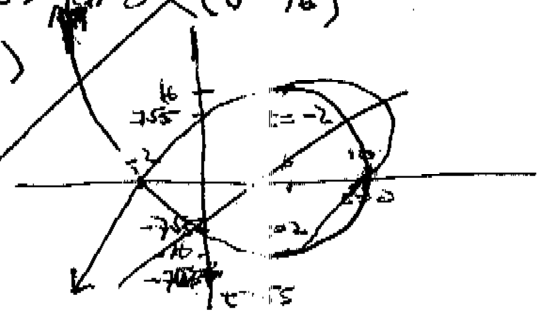
$x' = -2t$ and $y' = 3(t^2-4) = (t+2)(t-2)$

x-int @ $x = 10$ and $x = -2$ (double zero).

y-int @ when $t = \pm\sqrt{5} \Rightarrow y = 7\sqrt{5}$ ($t=\sqrt{5}$) and $y = 7\sqrt{5}$ ($t = -\sqrt{5}$)

horizontal @ $t = \pm 2 \Rightarrow (6, -16)$ ($t=2$) and $(6, 16)$ ($t=-2$)

vertical @ $t = 0 \Rightarrow (10, 0)$



<u>Ex1:</u> Sketch a graph of the curve parametrically given by

$x(t) = 10 - t^2$ and $y = t^3 - 12t$,

<u>x-intercepts:</u> $(y=0)$

solve $0 = t^3 - 12t$

$= t(t+\sqrt{12})(t-\sqrt{12})$.

$\Rightarrow t = 0$ or $t = \pm\sqrt{12}$

$t = 0 \Rightarrow (10,0)$

$t = \sqrt{12} \Rightarrow (-2,0)$

$t = -\sqrt{12} \Rightarrow (-2,0)$

<u>y-intercepts</u> $(x=0)$

solve $0 = 10 - t^2$

$\Rightarrow t = \pm\sqrt{10}$

$t = \sqrt{10} \Rightarrow (0, -2\sqrt{10})$

$t = -\sqrt{10} \Rightarrow (0, 2\sqrt{10})$

<u>slope undefined.</u> $(x'(t) = 0)$

solve $0 = -2t$

$\Rightarrow t = 0$

$t = 0 \Rightarrow (10,0)$

<u>slope = 0</u> $(y'(t) = 0)$

solve $0 = 3t^2 - 12$

$= 3(t^2 - 4)$

$\Rightarrow t = \pm 2$

$t = 2 \Rightarrow (6, -16)$

$t = -2 \Rightarrow (6, 16)$

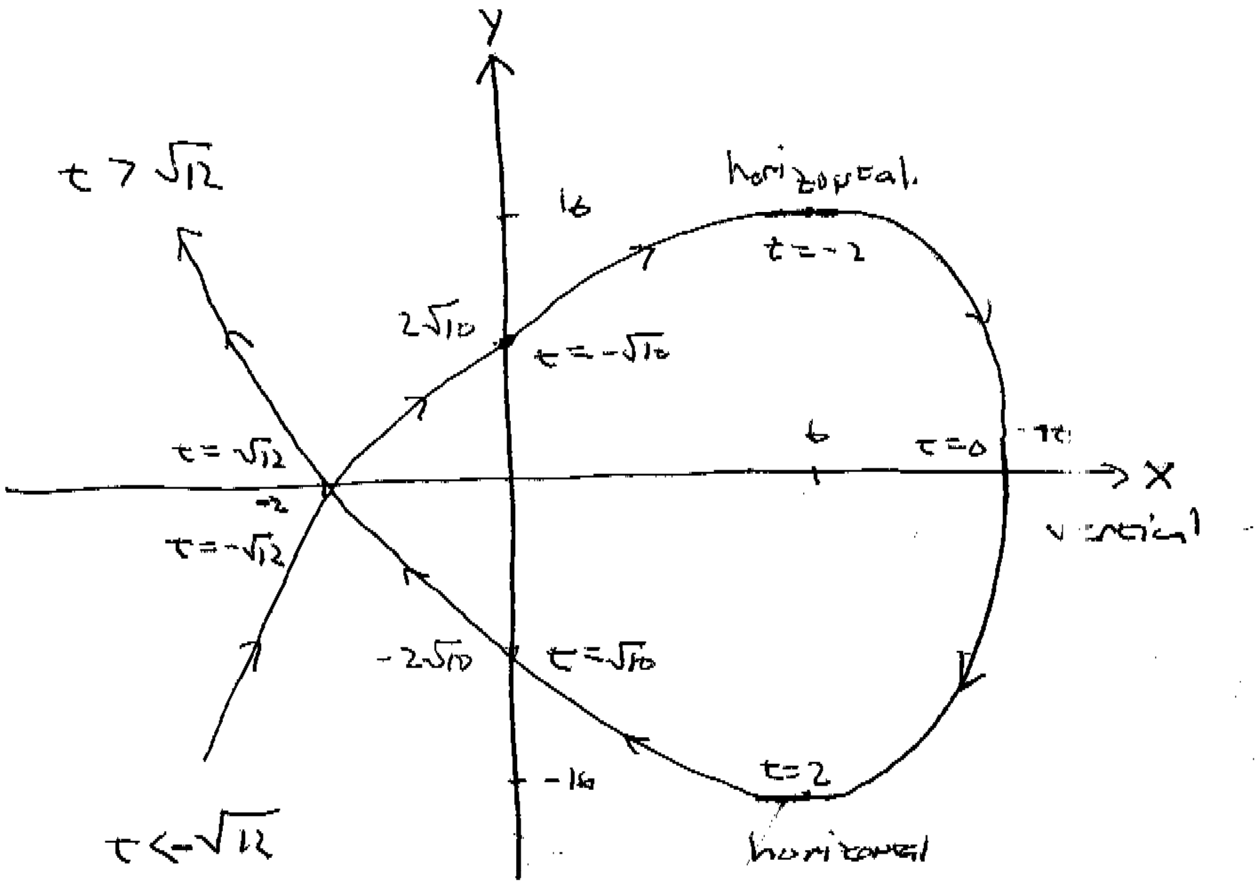
Find the slopes at $t = \pm\sqrt{12}$

$$\frac{dy}{dx} = \frac{3t^2 - 12}{-2t}$$

$$= -\frac{3}{2}\left(\frac{t^2-4}{t}\right)$$

$t = \sqrt{12} \Rightarrow m < 0$

$t = -\sqrt{12} \Rightarrow m > 0$



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The area "under" $(x(t), y(t))$ on $\alpha \le t \le \beta$ if the curve is traversed exactly once is given by

$$\boxed{3} \quad A = \int_{t=\alpha}^{t=\beta} y\,dx = \int_{t=\alpha}^{t=\beta} y(t)\,x'(t)\,dt.$$

($t=\alpha$ assumed to be on the left).

Ex 2: Find the area of a circle.

Let $x = \cos\theta$ and $y = \sin\theta$ on $0 \le \theta \le 2\pi$

$$A = \int_0^{2\pi} y(\theta)\,x'(\theta)\,d\theta$$

$$= \int_0^{2\pi} \sin(\theta)\,(-\sin(\theta))\,d\theta$$

$$= \int_{2\pi}^{0} \sin^2\theta\,d\theta$$

$$= \frac{1}{2}\int_{2\pi}^{0} (1 - \cos(2\theta))\,d\theta$$

$$= \frac{1}{2}\left[\theta - \frac{\sin(2\theta)}{2}\right]_{2\pi}^{0}$$

$= -\pi$ $\longleftarrow$ negative since $\theta = 0$ is right of $\theta = 2\pi$

For the right sign, $A = 2\int_{\pi}^{0} y(\theta)\,x'(\theta)\,d\theta$

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If a curve C w/pts $(x(t), y(t))$ on $\alpha \le t \le \beta$ where x' and y' are cont on $[\alpha, \beta]$ and C is traversed exactly once on $[\alpha, \beta]$, then the arclength of C is

$$\boxed{4} \quad L = \int_{\alpha}^{\beta} \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

Ex 3: Find the circumference of a circle w/radius r.

$x(\theta) = r\cos\theta$ and $y(\theta) = r\sin\theta$

$x'(\theta) = -r\sin\theta$ and $y'(\theta) = r\cos\theta$

$$L = \int_0^{2\pi} \sqrt{(-r\sin\theta)^2 + (r\cos\theta)^2}\, d\theta$$

$$= \int_0^{2\pi} r\sqrt{\sin^2\theta + \cos^2\theta}\, d\theta$$

$$= \int_0^{2\pi} r\, d\theta$$

$$= \Big[r\theta\Big]_0^{2\pi}$$

$$= 2\pi r$$

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If the curve given by $(x(t), y(t))$ on $\alpha \leq t \leq \beta$ is rotated about the x-axis, where x' and y' are cont and $y(t) \geq 0$ on $[\alpha, \beta]$, then the resulting surface area is.

$\boxed{5}$ $$SA = \int_{\alpha}^{\beta} 2\pi\, y(t) \sqrt{(x'(t))^2 + (y'(t))^2}\, dt$$

Ex 4: ~~Bor~~ Find the SA of the sphere w/ radius r.

$$SA = \int_0^{\pi} 2\pi r \sin\theta \, d\theta$$

$$= \Big[-2\pi r \cos\theta\Big]_0^{\pi}$$

$$= -2\pi r\,[-1 - (1)]$$

$$= 4\pi r \,//.$$