Mathematics: The Loss of Certainty
by Morris Kline
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carry any secret art but proceeds by quite definite and stateable rules which are the guarantee of the absolute objectivity of its judgment." Every mathematician, he said, shares the conviction that each definite mathematical problem must be capable of being solved. In his 1930 article, "Natural Knowledge and Logic," he said, "The real reason for Comte's failure to find an unsolvable problem is, in my opinion, that an unsolvable problem does not exist."

In "The Foundations of Mathematics," a paper read in 1927 and published in 1930, Hilbert elaborated on one he wrote in 1905. Referring to his metamathematical method (proof theory) of establishing consistency and completeness, he affirmed:

> With this new way of providing a foundation for mathematics, which we may appropriately call a proof theory, I pursue a significant goal, for I should like to eliminate once and for all the questions regarding the foundations of mathematics, in the form in which they are now posed, by turning every mathematical proposition into a formula that can be concretely exhibited and strictly derived, thus recasting mathematical derivations and inferences in such a way that they are unshakable and yet provide an adequate picture of the whole science. I believe I can attain this goal completely with my proof theory, even if a great deal of work must still be done before it is fully developed.

Clearly Hilbert was confident that his proof theory would settle the questions of consistency and completeness.

By 1930, several results on completeness had been obtained. Hilbert himself had constructed a somewhat artificial system which covered only a portion of arithmetic and had established its consistency and completeness. Other such limited results were soon obtained by other men. Thus relatively trivial axiomatic systems such as the propositional calculus were proven to be consistent and even complete. Some of these proofs were made by students of Hilbert. In 1930, Kurt Gödel (1906–1978), later a professor at the Institute for Advanced Study, proved the completeness of the first order predicate calculus which covers propositions and propositional functions.* These results delighted the formalists. Hilbert was confident that his metamathematics, his proof theory, would succeed in establishing consistency and completeness for all of mathematics.

Gödel

But in the very next year Gödel published another paper that opened up a Pandora's box. This paper, "On Formally Undecidable Propositions of *Principia Mathematica* and Related Systems" (1931), con-

*It is also consistent and its axioms are independent. This was shown by Hilbert and other men.

tained two startling results. To the mathematical world, the more devastating assertion was that the consistency of any mathematical system that is extensive enough to embrace even the arithmetic of whole numbers cannot be established by the logical principles adopted by the several foundational schools, the logicists, the formalists, and the set-theorists. This applies especially to the formalist school because Hilbert had deliberately limited his metamathematical logical principles to those acceptable even to the intuitionists and so fewer logical tools were available. This result prompted Hermann Weyl to say that God exists because mathematics is undoubtedly consistent and the devil exists because we cannot prove the consistency.

The above result of Gödel's is a corollary of his other equally startling result, which is called Gödel's incompleteness theorem. It states that, if any formal theory *T* adequate to embrace the theory of whole numbers is consistent, then *T* is incomplete.* This means that there is a meaningful statement of number theory, let us call it *S*, such that neither *S* nor not *S* is provable in the theory. Now either *S* or not *S* is true; there is, then, a true statement of number theory which is not provable and so is undecidable. Though Gödel was not too clear on the class of axiom systems involved, his theorem does apply to the Russell-Whitehead system, the Zermelo-Fraenkel system, Hilbert's axiomatization of number theory, and in fact to all widely accepted axiom systems. Apparently the price of consistency is incompleteness. To add insult to injury, some of the undecidable statements can be shown to be true by arguments, that is, rules of reasoning, that transcend the logic used in the formal systems just mentioned.

As one might expect, Gödel did not obtain his amazing results readily. His overall scheme was to associate numbers with each symbol and sequence of symbols of, for example, the logistic and formalistic approaches to mathematics. Then to any proposition or set of propositions constituting a proof, he also attached a Gödel number.

Specifically his arithmetization consisted in assigning natural numbers to mathematical concepts. 1 is assigned 1. To the equals sign, he assigned 2; to Hilbert's negation symbol, he assigned 3; to the plus sign he assigned 5, and similarly for the other symbols. Thus for the collection of symbols $1=1$, he had the number symbols 1, 2, 1. However, Gödel now assigned to the formula $1=1$, not the symbols 1, 2, 1, but a single number which nevertheless still showed the component numbers. He took the first three prime numbers, 2, 3, 5,

* This result applies also to the second order predicate calculus (Chapter VIII). Incompleteness does not invalidate those theorems that can be proved.

and formed $2^1 \cdot 3^2 \cdot 5^1 = 90$. So to $1=1$ he assigned the number 90. Note that 90 can always be decomposed uniquely to $2^1 \cdot 3^2 \cdot 5^1$, so that we can recover the symbols 1, 2, 1.

To each formula of the systems he considered, Gödel assigned a number. And to an entire sequence of formulas which constitute a proof, he likewise assigned a number. The exponents of such a number are the numbers of formulas. They are not themselves prime but are attached to primes. Thus $2^{900} \cdot 3^{90}$ can be the number of a proof. This proof contains the formula 900 and the formula 90. Hence from the number of a proof, we can reconstruct the formulas of the proof.

Then Gödel showed that the concepts of metamathematics about the formulas of the formal systems he considered are also representable by numbers. Thereby each assertion in metamathematics has a Gödel number assigned to it. This number is the number of a metamathematical statement. It is also a number of some arithmetical statement. Thus metamathematics is also "mapped" into arithmetic.

In these arithmetical terms Gödel showed how to construct an *arithmetical assertion* G that says, in the verbal metamathematical language, that the statement with Gödel number m, say, is not provable. But G, as a sequence of symbols, has the Gödel number m. Thus, G says of itself that it is not provable. But if the entire *arithmetical assertion* G is provable, it asserts that it is not provable, and if G is not provable it affirms just that and so is not provable. However, since the arithmetical assertion is either provable or not provable, the formal system to which the arithmetical assertion belongs, if *consistent*, is incomplete. Nevertheless, the arithmetical statement G is true because it is a statement about integers that can be established by more intuitive reasoning than the formal systems permit.

The essence of Gödel's scheme may also be seen from the following example. If one considers the statement, "This sentence is not true," we have a contradiction. For if the entire sentence is true, then, as it asserts, it is false; and if the entire sentence is false, then it is true. Gödel substituted unprovable for false so that the sentence reads, "This sentence is unprovable." Now if the statement is not provable, then what it says is true. If, on the other hand, the sentence is provable, it is not true or, by standard logic, if true, it is not provable. Hence the sentence is true if and only if it is *not* provable. Thus the result is not a contradiction but a true statement which is unprovable or undecidable.

After exhibiting his undecidable statement Gödel constructed an arithmetical statement A that represents the metamathematical statement "Arithmetic is consistent," and he proved that A implies G. Hence if A were provable, G would be provable. But since G is undecidable, A is not provable. It is undecidable. This result establishes the impossibility of proving consistency by any method or set of logical principles that can be translated into the system of arithmetic.

It would seem that incompleteness could be avoided by adding to the logical principles or by adding a mathematical axiom to the formal system. But Gödel's method shows that, if the additional statement is also expressible in arithmetical terms by his scheme of assigning numbers to the symbols and formulas, then an undecidable statement can still be formulated. Put otherwise, undecidable statements can be avoided and consistency proved only by means of principles of reasoning that cannot be "mapped" into arithmetic. To use a somewhat loose analogy, if the principles of reasoning and mathematical axioms were in Japanese and the Gödel arithmetization were in English, then as long as the Japanese could be translated into English, the Gödel results would obtain.

Thus Gödel's incompleteness theorem asserts that no system of mathematical and logical axioms that can be arithmetized in some manner such as Gödel used is adequate to encompass all the truths of even that one system, to say nothing about all of mathematics, because any such axiom system is incomplete. There exist meaningful statements that belong to these systems but cannot be proved within the systems. They can nevertheless be shown to be true by non-formal arguments. This result, that there are limitations on what can be achieved by axiomatization, contrasts sharply with the late 19th-century view that mathematics is coextensive with the collection of axiomatized branches. Gödel's result dealt a death blow to comprehensive axiomatization. This inadequacy of the axiomatic method is not in itself a contradiction, but it is surprising because mathematicians, the formalists in particular, had expected that any true statement could certainly be established within the framework of some axiomatic system. Thus, while Brouwer made clear that what is intuitively certain falls short of what is proved in classical mathematics, Gödel showed that what is intuitively certain extends beyond mathematical proof. As Paul Bernays has said, it is less wise today to recommend axiomatics than to warn against an overevaluation of it. Of course, the above arguments do not exclude the possibility that new methods of proof may go beyond what the logical principles accepted by the foundational schools permit.

Both of Gödel's results were shattering. The inability to prove consistency dealt a death blow most directly to Hilbert's formalist philosophy because he had planned such a proof in his metamathematics and was confident it would succeed. However, the disaster extended far beyond Hilbert's program. Gödel's result on consistency says that we cannot prove consistency in any approach to mathematics by safe logical principles. No one of the approaches that had been put forth was excepted. The one distinguishing feature of mathematics that it might have claimed in this century, the absolute certainty or validity of its results, could no longer be claimed. Worse, since consistency cannot be proved, mathematicians risked talking nonsense because any day a contradiction could be found. If this should happen and the contradiction were not resolvable, then all of mathematics would be pointless. For, of two contradictory propositions, one must be false, and the logical concept of implication adopted by all the mathematical logicians, called material implication (Chapter VIII), allows a false proposition to imply any proposition. Hence mathematicians were working under a threat of doom. The incompleteness theorem was another blow. Here, too, Hilbert was directly involved, but the theorem applies to all formal approaches to mathematics.

Though mathematicians generally had not expressed themselves so confidently as Hilbert had, they certainly expected to solve any clear-cut problem. Indeed, the efforts to prove, for example, Fermat's last "theorem," which asserts that there are no integers that satisfy $x^n+y^n=z^n$ when n is greater than 2, had even by 1930 produced hundreds of long and deep papers. Perhaps all of these efforts were in vain because the assertion may very well be undecidable.

Gödel's incompleteness theorem is to an extent a denial of the law of excluded middle. We believe a proposition is true or false, and in modern foundations this means provable or disprovable by the laws of logic and any axioms of the particular subject to which the proposition belongs. But Gödel showed that some are neither provable or disprovable. This is an argument for the intuitionists who argued against the laws on other grounds.

There is a possibility of proving consistency if one can show, unlike Gödel's approach, that a system contains an undecidable proposition, for, by the argument noted earlier, concerning material implication, if there were a contradiction, every proposition could be proved. But thus far this has not been done.

Hilbert was not convinced that he had failed. He was an optimist. He had unbounded confidence in the power of man's reason and understanding. This optimism gave him courage and strength, but it barred him from granting that there could be undecidable mathematical problems. For Hilbert, mathematics was a domain in which the researcher could find no bounds other than his own personal power.

Gödel's 1931 results were published between the writing of the first volume (actually published in 1934) and the second volume (1939) of a basic work on foundations by Hilbert and Paul Bernays. Hence in the preface to the second volume the authors agreed that one must enlarge the methods of reasoning in metamathematics. They included trans-