What is Mathematics, Really?

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they couldn't *be* cubes. No ice cube or ebony cube or brass cube has 12 edges all *exactly* the same length, 8 corners all *perfectly* square.

Only a mathematical 3-cube is a perfect cube. So a mathematical 3-cube is like a 4-cube, in not being a physical object! Then what is it? Where is it? Is there a big difference between asking, "Does a 3-cube exist?" and asking, "Does a 4-cube exist?"

Answering these questions is the point of this book.

For experts, here are some exercises in the philosophy of mathematics that anticipate much of what follows.

How would these questions about 3-cube and 4-cube be answered by Gödel or Thom? (See below, "Must We Be Platonists?")

By Frege or Russell in his logicist period?

By Brouwer or Bishop?

By Hilbert or Bourbaki (Chapter 8)?

By Wittgenstein (Chapter 11)?

By Quine? By Putnam in his phase I, II, or III (Chapter 9)?

Quick Overview

Even without three years of graduate school, you can get a rough notion of modern mathematics. Here's a mini-sketch of its method and matter.

The method of mathematics is "conjecture and proof." You come to an inherited network of concepts and facts, properties and connections, called a "theory." (For instance, classical solid geometry, including the 3-cube.) This presently existing theory is the result of a historic evolution. It is the cooperative and competitive work of generations of mathematicians, associated by friendship and rivalry, by mutual criticism and correction, as leaders and followers, mentors and protégés.

Starting with the theory as you find it, you fill in gaps, connect to other theories, and spin out enlargements and continuations—like going up one dimension to dream of a "hypercube."

You just solved the hypercube problem. But you didn't solve it in isolation. You were handed the problem in the first place. Then you got helpful hints and encouragement as you went along. When you finally got the answer, you received confirmation that your answer was right.

Believe it or not, a mathematician has needs similar to yours. He/she needs to discover a problem connected to the existing mathematical culture. Then she needs reassurance and encouragement as she struggles with it. And in the end when she proposes a solution she needs agreement or criticism. No matter how isolated and self-sufficient a mathematician may be, the source and verification of his work goes back to the community of mathematicians.

Sometimes new theories seem to spin out of your head and the heads of your predecessors. Sometimes they're suggested by real-world subjects, like physics. Today the infinite-dimensional spaces of higher geometry are models for the elementary particles of physics.

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Mathematical discovery rests on a validation called "proof," the analogue of experiment in physical science. A proof is a conclusive argument that a proposed result follows from accepted theory. "Follows" means the argument convinces qualified, skeptical mathematicians. Here I am giving an overtly social definition of "proof." Such a definition is unconventional, yet it is plainly true to life.

In logic texts and modern philosophy, "follows" is often given a much stricter sense, the sense of mechanical computation. No one says the proofs that mathematicians write actually *are* checkable by machine. But it's conventional to insist that there be *no doubt* they *could* be checked that way.

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Such lofty rigor isn't found in all mathematics. From one specialty to another, from one mathematician to another, there's variation in strictness of proof and applicability of results. Mathematics that stresses results above proof is often called "applied mathematics." Mathematics that stresses proof above results is sometimes called "pure mathematics," more often just "mathematics." (Outsiders sometimes say "theoretical mathematics.")

A naive non-mathematician—perhaps a neo-Fregean analytic philosopher—looks into Euclid, or a more modern math text of formalist stripe, and observes that axioms come first. They're right on page one. He or she understandably concludes that in mathematics, axioms come first. First your assumptions, then your conclusions, no?

But anyone who has done mathematics knows what comes first—a problem. Mathematics is a vast network of interconnected problems and solutions. Sometimes a problem is called "a conjecture."

Sometimes a solution is a set of axioms!

I explain.

When a piece of mathematics gets big and complicated, we may want to systematize and organize it, for esthetics and for convenience. The way we do that is to axiomatize it. Thus a new type of problem (or "meta-problem") arises:

"Given some specific mathematical subject, to find an attractive set of axioms from which the facts of the subject can conveniently be derived."

Any proposed axiom set is a proposed solution to this problem. The solution will not be unique. There's a history of re-axiomatizations of Euclidean geometry, from Hilbert to Veblen to Birkhoff the Elder.

In developing and understanding a subject, axioms come late. Then in the formal presentations, they come early.

Sometimes someone tries to invent a new branch of mathematics by making up some axioms and going from there. Such efforts rarely achieve recognition or permanence. Examples, problems, and solutions come first. Later come axiom sets on which the already existing theory can be "based."

The view that mathematics is in essence derivations from axioms is backward. In fact, it's wrong.

An indispensable partner to proof is mathematical intuition. This tells us what to try to prove. We relied heavily on intuition in our hypercube exercise. It often gives true theorems, even with gappy proofs. We return to intuition and proof in Chapter 4.

So far I've described mathematics by its methods. What about its content? The dictionary says math is the science of number and figure ("figure" meaning the shapes or figures of geometry.) This definition might have been O.K. 200 years ago. Today, however, math includes the groups, rings, and fields of abstract algebra, the convergence structures of point-set topology, the random variables and martingales of probability and mathematical statistics, and much, much more. *Mathematical Reviews* lists 3,400 subfields of mathematics! No one could attempt even a brief presentation of all 3,400, let alone a philosophical investigation of them all. To identify a branch of study as part of mathematics, one is guided by its method more than its content.

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method over content.

Formalism: A First Look

Two principal views of the nature of mathematics are prevalent among mathematicians—Platonism and formalism. Platonism is dominant, but it's hard to talk about it in public. Formalism feels more respectable philosophically, but it's almost impossible for a working mathematician to really believe it.

The next section is about Platonism. Here I take a quick glance at formalism. I return to it in the section of Chapter 9 on David Hilbert. The third major school, intuitionism or constructivism, is also discussed in Chapter 9, in the sections on foundations, on L. E. J. Brouwer, and on Errett Bishop.

The formalist philosophy of mathematics is often condensed to a short slogan: "Mathematics is a meaningless game." ("Meaningless" and "game" remain undefined. Wittgenstein showed that games have no strict definition, only a family resemblance.)

What do formalists mean by "game" when they call mathematics a game? Perhaps they use "game" to mean something "played by the rules." (Now "play" and "rule" are undefined!)

For a game in that sense, two things are needed:

(2) people to play by the rules.
(1) rules.

Rule-making can be deliberate, as in Monopoly or Scrabble—or spontaneous, as in natural languages or elementary arithmetic.

In either case, the *making* of rules doesn't follow rules!

Wittgenstein and some others seem to think that since the making of rules doesn't follow rules, then the rules are arbitrary. They could just as well be any way at all. This is a gross error.

The rules of language and of mathematics are historically determined by the workings of society that evolve under pressure of the inner workings and interactions of social groups, and the physical and biological environment of earth. They are also simultaneously determined by the biological properties, especially the nervous systems, of individual humans. Those biological properties and nervous systems have permitted us to evolve and survive on earth, so of course they reflect somehow the physical and biological properties of this planet. Complicated, certainly. Mysterious, no doubt. Arbitrary, no.

People often make rules deliberately. Not only for games, but also for computer languages, for parliamentary procedure, for stopping at STOP signs, and for Orthodox weddings. These rule-making tasks don't follow rules. But that doesn't make them arbitrary. Rules are made for a purpose. To be played or accepted or performed by people, they have to be playable or acceptable by people. Tradition, taste, judgment, and consensus matter. Eccentricities of individual rule-makers matter. The resultant of such social and personal factors is what makes us make the rules we make. The outcome of rule-making isn't arbitrary. Neither is it rule governed.

Some details of a rule system may seem arbitrary or optional. In chess, for instance, the rule for castling might be varied without ruining the game.

Is there a sharp separation between playing by the rules and making the rules?

Some formalists in philosophy of mathematics say discovery is lawless—has no logic—while proof or justification is nothing but logic. If such a philosopher notices that real mathematical life isn't that way, the discrepancy seems like a scandal that must be kept out of the newspapers, or a crime calling for correction by Georg Kreisel's "logical hygiene."

In real life, in all games including mathematics (supposing for the moment that it is a game), the separation between playing the game and making the rules is imperfect, partial, incomplete.

Chess players don't change the rules of chess as they go along. Not in tournament chess, at any rate. Disputes are settled according to written procedures. But these procedures aren't *rules*. Settling disputes comes down to judgments and opinions. Big league baseball has plenty of rules. But the game would be impossible without umpires who use their judgment. In street stick ball, first base is supposed to be the left front fender on the closest car parked on the right side of the street. If no car is there, we improvise.

In real life there are no totally rule-governed activities. Only more or less rule-governed ones, with more or less definite procedures for disputes. The rules and procedures evolve, sometimes formally like amending the U.S. Constitution, sometimes informally, as street games evolve with time and mixing of cultures. Is there totally unruly or ruleless behavior? Perhaps not. Mathematics is in part a rule-governed game. But one can't overlook how the rules are made, how they evolve, and how disputes are resolved. That isn't rule governed, and can't be.

Computer proof is changing the way the game of mathematics is played. Wolfgang Haken thinks computer proof is permitted under the rules. Paul Halmos thinks it ought to be against the rules. Tom Tymoczko thinks it amounts to changing the rules. In the long run, what mathematicians publish, cite, and especially teach, will decide the rules. We have no French Academy to set rules, no cabal of team owners to say how to play our game. Our rules are set by our consensus, influenced and led by our most powerful or prestigious members (of course).

These considerations on games and rules in general show that one can't understand mathematics (or any other nontrivial human activity) by simply finding rules that it follows or ought to follow. Even if that could be done, it would lead to more interesting questions: Why and whence those rules?

The notion of strictly following rules without any need for judgment is a fiction. It has its use and interest. It's misleading to apply it literally to real life.

Must We Be Platonists?

Platonism, or realism as it's been called, is the most pervasive philosophy of mathematics. It has various variations. The standard version says mathematical entities exist outside space and time, outside thought and matter, in an abstract realm independent of any consciousness, individual or social. Today's mathematical Platonisms descend in a clear line from the doctrine of Ideas in Plato (see "Plato" in Chapter 6). Plato's philosophy of mathematics came from the Pythagoreans, so mathematical "Platonism" ought to be "Pythago-Platonism." I defer to custom and say "Platonism." (This debt of Plato is discussed by John Dewey in his 1929 Gifford lectures and by Bertrand Russell in Chapter 9.)

There are Platonisms of mathematicians and Platonisms of philosophers. I quote half a dozen eminent Platonists of past and present, mostly mathematicians. (Somerville and Everett are copied from Leslie White's article in *The World of Mathematics.*)

Edward Everett (1794–1865), the first American to receive a doctorate at Göttingen, an orator who shared the platform with Abraham Lincoln at Gettysburg, wrote: "In the pure mathematics we contemplate absolute truths which existed in the divine mind before the morning stars sang together, and which will continue to exist there when the last of their radiant host shall have fallen from heaven."

The scholar and mathematician Mary Somerville (1780–1872): "Nothing has afforded me so convincing a proof of the unity of the Deity as these purely mental conceptions of numerical and mathematical science which have been by slow degrees vouchsafed to man, and are still granted in these latter times by the Differential Calculus, now superseded by the Higher Algebra, all of which must have existed in that sublimely omniscient Mind from eternity."

G. H. Hardy, the leading English mathematician of the 1920s: "I have myself always thought of a mathematician as in the first instance an observer, who gazes at a distant range of mountains and notes down his observations. His object is simply to distinguish clearly and notify to others as many different peaks as he can. There are some peaks which he can distinguish easily, while others are less clear. He sees A sharply, while of B he can obtain only transitory glimpses. At last he makes out a ridge which leads from A and, following it to its end, he discovers that it culminates in B. B is now fixed in his vision, and from this point he can proceed to further discoveries. In other cases perhaps he can distinguish a ridge which vanishes in the distance, and conjectures that it leads to a peak in the clouds or below the horizon. But when he sees a peak, he believes that it is there simply because he sees it. If he wishes someone else to see it, he *points to it*, either directly or through the chain of summits which led him to recognize it himself. When his pupil also sees it, the research, the argument, the *proof* is finished" (1929, p. 18). Here the "chain of summits" is the chain of statements in a proof, connecting known facts (peaks) to new ones. Hardy uses a chain of summits to find a new peak. Once he sees the new peak, he believes in it because he sees it, no longer needing any chain.

The preeminent logician, Kurt Gödel: "Despite their remoteness from sense experience, we do have something like a perception also of the objects of set theory, as is seen from the fact that the axioms force themselves upon us as being true. I don't see any reason why we should have less confidence in this kind of perception, i.e., in mathematical intuition, than in sense perception. . . . This, too, may represent an aspect of objective reality."

The French geometer and Fields Medalist René Thom, father of catastrophe theory: "Mathematicians should have the courage of their most profound convictions and thus affirm that mathematical forms indeed have an existence that is independent of the mind considering them. . . . Yet, at any given moment, mathematicians have only an incomplete and fragmentary view of this world of ideas."

Thom's world of ideas is geometric; Gödel's is set-theoretic. They believe in an independent world of ideas—but not the same world!

Paul Erdös was a famous Hungarian mathematician who talked about "The Book." "The Book" contains all the most elegant mathematical proofs, the known and especially the unknown. It belongs to "the S. F."—"the Supreme Fascist"—Erdös's pet name for the Almighty. Occasionally the S. F. permits someone a quick glimpse into the Book.

The Book is a perfect metaphor for Platonism. But Erdös said he's not interested in philosophy. The Book and the S. F. are "only a joke."

However, in a film about Erdös (*N is a Number*, produced by Paul Csicsery) his friend and collaborator Fam Chung, says, "In Paul's mind there is only one reality, and that's mathematics."

Ron Graham, a well-known combinatorialist, collaborator friend of Erdös and husband of Chung, goes even further: "I personally feel that mathematics is the essence of what's driving the universe."

Another Erdös collaborator, Joel Spencer: "Where else do you have absolute truth? You have it in mathematics and you have it in religion, at least for some people. But in mathematics you can really argue that this is as close to absolute truth as you can get. When Euclid showed that there were an infinite number of primes, *that's it!*. There are an infinite number of primes, no ifs, ands, or buts! That's as close to absolute truth as I can see getting."

(As a small point of historical fidelity, Euclid never could have said there was an infinite number of anything. Proposition 20, Book IX, says, in Heath's translation, "Prime numbers are more than any assigned multitude of prime numbers"—there is no greatest prime. Heath immediately paraphrases this as "the important proposition that the number of prime numbers is infinite." Heath's and Spencer's formulation is natural in today's context of infinite sets. Not in Euclid's context.)

Why do mathematicians believe something so unscientific, so far-fetched as an independent immaterial timeless world of mathematical truth? Why be a Platonist?

The mystery of mathematics is its objectivity, its seeming certainty or near-certainty, and its near-independence of persons, cultures, and historical epochs (see the section on Change in Chapter 5).

Platonism says mathematical objects are real and independent of our knowledge. Space-filling curves, uncountably infinite sets, infinite-dimensional manifolds—all the members of the mathematical zoo—are definite objects, with definite properties, known or unknown. These objects exist outside physical space and time. They were never created. They never change. By logic's law of the excluded middle, a meaningful question about any of them has an answer, whether we know it or not. According to Platonism a mathematician is an empirical scientist, like a botanist. He can't invent, because everything is already true? there. He can only discover. Our mathematical knowledge is objective and unchanging because it's knowledge of objects external to us, independent of us, which are indeed changeless.

An inarticulate, half-conscious Platonism is nearly universal among mathematicians. Research or problem-solving, even at the elementary level, generates a naive, uncritical Platonism. In math class, everybody has to get the same answer. Except for a few laggards, they *do* all get the same answer! That's what's special about math. *There are right answers.* Not right because that's what Teacher wants us to believe. Right because *they are* right.

That universality, that independence of individuals, makes mathematics seem immaterial, inhuman. Platonism of the ordinary mathematician or student is a recognition that the facts of mathematics are independent of her or his wishes. This is the quality that makes mathematics exceptional.

Yet most of this Platonism is half-hearted, shamefaced. We don't ask, How does this immaterial realm relate to material reality? How does it make contact with flesh and blood mathematicians? We refuse to face this embarrassment:

Ideal entities independent of human consciousness violate the empiricism of modern science. For Plato the Ideals, including numbers, are visible or tangible in Heaven, which we had to leave in order to be born. For Leibniz and Berkeley, abstractions like numbers are thoughts in the mind of God. That Divine Mind is still real for Somerville and Everett.

Heaven and the Mind of God are no longer heard of in academic discourse. Yet most mathematicians and philosophers of mathematics continue to believe in an independent, immaterial abstract world—a remnant of Plato's Heaven, attenuated, purified, bleached, with all entities but the mathematical expelled.

Can't have Platonism w/o God.

Platonism without God is like the grin on Lewis Carroll's Cheshire cat. The cat had a grin. Gradually the cat disappeared, until all was gone—except the grin. The grin remained without the cat.

MacLane is unusual in his unequivocal rejection of Platonism, without turning to formalism. "The platonic notion that there is somewhere the ideal realm of sets, not yet fully described, is a glorious illusion" (p. 385). He thinks there's no need to consider the question of existence of mathematical entities.

The Platonisms of philosophers are more sophisticated than those of mathematicians. One of them is logicism, once preached by Gottlob Frege and Bertrand Russell. Today's "most influential philosopher," W. V. O. Quine, has his own pragmatic-type Platonism (see Chapter 9). Here we talk mainly about "garden variety" or "generic" Platonism, Platonism among the broad mathematical masses.

The objections to Platonism are never answered: the strange parallel existence of two realities—physical and mathematical; and the impossibility of contact between the flesh-and-blood mathematician and the immaterial mathematical object. Platonism shares the fatal flaw of Cartesian dualism. To explain the existence and properties of mind and matter, Descartes postulated a different "substance" for each. But he couldn't plausibly explain how the two substances interact, as mind and body do interact. In similar fashion, Platonists explain mathematics by a separate universe of abstract objects, independent of the material universe. But how do the abstract and material universes interact? How do flesh-and-blood mathematicians acquire the knowledge of number?

To answer, you have to forget Platonism, and look in the socio-cultural past and present, in the history of mathematics, including the tragic life of Georg Cantor.

The set-theoretic universe constructed by Cantor and generally adopted by Platonists is believed to include all mathematics, past, present, and future. In it, the uncountable set of real numbers is just the beginning of uncountable chains of uncountables. The cardinality of this set universe is unspeakably greater than that of the material world. It dwarfs the material universe to a tiny speck. And it was all there before there was an earth, a moon, or a sun, even before the Big Bang. Yet this tremendous reality is unnoticed! Humanity dreams on, totally unaware of it—*except for us mathematicians.* We alone notice it. But only since Cantor revealed it in 1890. Is this plausible? Is this credible? Roger Penrose declares himself a Platonist, but draws the line at swallowing the whole set-theoretic hierarchy.

Platonists don't acknowledge the arguments against Platonism. They just re-avow Platonism.

Frege's point of view persists today among set-theoretic Platonists. It goes something like this:

1. Surely the empty set exists—we all have encountered it!
2. Starting from the empty set, perform a few natural operations, like forming the set of all subsets. Before long you have a magnificent structure in which you can embed the real numbers, complex numbers, quaternions, Hilbert spaces, infinite-dimensional differentiable manifolds, and anything else you like.
3. Therefore it's vain to talk of inventing or creating mathematics. In this all-encompassing, set-theoretic structure, everything we could ever want or dream of is already present.

Yet most advances in mainstream mathematics are made without reference to any set-theoretic embedding. Saying Hilbert space was already there in the set universe is like telling Rodin, "*The Thinker* is a nice piece of work, but all you did was get rid of the extra marble. The statue was there inside the marble quarry before you were born."

Rodin made *The Thinker* by removing marble. Hilbert, von Neumann, and the rest made the theory of Hilbert space by analyzing, generalizing, and rearranging mathematical ideas that were present in the mathematical atmosphere of their time.

A Way Out

What's the nature of mathematical objects?

The question is made difficult by a centuries-old assumption of Western philosophy: "There are two kinds of things in the world. What isn't physical is mental; what isn't mental is physical."

Mental is individual consciousness. It includes private thoughts—mathematical and philosophical, for example—before they're communicated to the world and become social—and also perception, fear, desire, despair, hope, and so on.

Physical is taking up space—having weight or energy. It's flesh and bones, sound waves, X-rays, galaxies.

Frege showed that mathematical objects are neither physical nor mental. He labeled them "abstract objects." What did he tell us about abstract objects? Only this: They're neither physical nor mental. abstract.

Are there other things besides numbers that aren't mental or physical?

Yes! Sonatas. Prices. Eviction notices. Declarations of war.

Not mental or physical, but not abstract either!

The U.S. Supreme Court exists. It can condemn you to death!

Is the Court physical? If the Court building were blown up and the justices moved to the Pentagon, the Court would go on. Is it mental? If all nine justices expired in a suicide cult, they'd be replaced. The Court would go on.

The Court isn't the stones of its building, nor is it anyone's minds and bodies. Physical and mental embodiment are necessary to it, but they're not *it*. *It's a social institution*. Mental and physical categories are insufficient to understand it. It's comprehensible only in the context of American society.

What matters to people nowadays?

Marriage, divorce, child care.
Advertising and shopping.
Jobs, salaries, money.
The news, and other television entertainment.
War and peace.

All these entities have mental and physical aspects, but none is a mental or a physical entity. Every one is a social entity.

Social reality distinct from physical and mental reality was explained by Émile Durkheim a century ago. These quotations are taken from an essay by L. White.

"Collective ways of acting and thinking have a reality outside the individuals who, at every moment of time, conform to it. These ways of thinking and acting exist in their own right. The individual finds them already formed, and he cannot act as if they did not exist or were different from how they are. . . . Of course, the individual plays a role in their genesis. But for a social fact to exist, several individuals, at the very least, must have contributed their action; and it is this combined action which has created a new product. Since this synthesis takes place outside each one of us (for a plurality of consciousness enters into it), its necessary effect is to fix, to institute outside us, certain ways of acting and certain judgments which do not depend on each particular will taken separately" (1938, p. 56).

"There are two classes of states of consciousness that differ from each other in origin and nature, and in the end toward which they aim. One class merely expresses our organisms and the object to which they are most directly related. Strictly individual, the states of consciousness of this class connect us only with ourselves, and we can no more detach them from us than we can detach ourselves from our bodies. The states of consciousness of the other class, on the contrary, come to us from society; they transfer society into us and connect us with something that surpasses us. Being collective, they are impersonal; they turn us toward ends that we hold in common with other men; it is through them and them alone that we can communicate with others. . . . In brief, this duality corresponds to the double existence that we lead concurrently: the one purely individual and rooted in our organism, the other social and nothing but an extension of society" (1964, p. 337).

Concepts have their own life, said Durkheim. "When once born they obey laws all their own. They attract each other, repel each other, unite, divide themselves and multiply" (1976, p. 424).

Mathematics consists of concepts. Not pencil or chalk marks, not physical triangles or physical sets, but concepts, which may be suggested or represented by physical objects.

In reviewing *The Mathematical Experience*, the mathematical expositor and journalist Martin Gardner made this objection: When two dinosaurs wandered to the water hole in the Jurassic era and met another pair of dinosaurs happily sloshing, there were four dinosaurs at the water hole, even though no human was present to think, "$2 + 2 = 4$." This shows, says Gardner, that $2 + 2$ really is 4 in reality, not just in some cultural consciousness. $2 + 2 = 4$ is a law of nature, he says, independent of human thought.

To untangle this knot, we must see that "2" plays two linguistic roles. Sometimes it's an adjective; sometimes it's a noun.

In "two dinosaurs," "two" is a *collective adjective*. "Two dinosaurs plus two dinosaurs equals four dinosaurs" is telling about dinosaurs. If I say "Two discrete, reasonably permanent, noninteracting objects collected with two others makes four such objects," I'm telling part of what's meant by discrete, reasonably permanent noninteracting objects. That is a statement in elementary physics.

John Stuart Mill pointed out that with regard to discrete, reasonably permanent non-interacting objects, experience tells us

$$2 + 2 = 4.$$

In contrast, "Two is prime but four is composite" is a statement about the pure numbers of elementary arithmetic. Now "two" and "four" are *nouns*, not adjectives. They stand for pure numbers, which are concepts and objects. They are *conceptual objects*, shared by everyone who knows elementary arithmetic, described by familiar axioms and theorems.

The collective adjectives or "counting numbers" are finite. There's a limit to how high anyone will ever count. Yet there isn't any last counting number. If you counted up to, say, a billion, then you could count to a billion and one. In pure arithmetic, these two properties—finiteness, and not having a last—are contradictory. This shows that the counting numbers aren't the pure numbers.

Consider the pure number $10^{(10^{10})}$. We easily ascertain some of its properties, such as: "The only prime factors of $10^{(10^{10})}$ are 2 and 5." But we can't count that high. In that sense, there's no counting number equal to $10^{(10^{10})}$.

Körner made the same distinction, using uppercase for Counting Numbers (adjectives) and lowercase for "pure" natural numbers (nouns). Jacob Klein wrote that a related distinction was made by the Greeks, using their words "arithmos" and "logistiké."

So "two" and "four" have double meanings: as Counting Numbers or as pure numbers. The formula

counting
pure #'s

$$2 + 2 = 4$$

has a double meaning. It's about counting—about how discrete, reasonably permanent, noninteracting objects behave. And it's a theorem in pure arithmetic (Peano arithmetic if you like). This linguistic ambiguity blurs the difference between Counting Numbers and pure natural numbers. But it's convenient. It's comparable to the ambiguity of nonmathematical words, such as "art" or "America."

The pure numbers rise out of the Counting Numbers. In a process related to Aristotle's abstraction, they disconnect from "real" objects, to exist as shared concepts in the mind/brains of people who know elementary arithmetic. In that realm of shared concepts, $2 + 2 = 4$ is a different fact, with a different meaning. And we can now show that it follows logically from other shared concepts, which we usually call axioms.

Platonist philosophy masks this social mode of existence with a myth of "abstract concepts."

From living experience we know two facts:

Bold claim.

Some ?

Fact 1: Mathematical objects are created by humans. Not arbitrarily, but from activity with existing mathematical objects, and from the needs of science and daily life.

Fact 2: Once created, mathematical objects can have properties that are difficult for us to discover. This is just saying there are mathematical problems which are difficult to solve. Example: Define x as the 200th digit in the decimal expansion of $23^{(45^{6789})}$. x is thereby determined. Yet I have no effective way to find it.

These two facts aren't theses waiting to be established! They're experiences needing to be understood. We need to "unpack" their philosophical consequences and their paradoxes.

Once created and communicated, mathematical objects are *there*. They detach from their originator and become part of human culture. We learn of them as external objects, with known properties and unknown properties. Of the unknown properties, there are some we are able to discover. Some we can't discover, even though they are our own creations. Does this sound paradoxical? If so, it's because of thinking that recognizes only two realities: the individual subject (the isolated interior life), and the exterior physical world. The existence of mathematics shows the inadequacy of those two categories. The customs, traditions, and institutions of our society are real, yet they are neither in the private inner nor the nonhuman outer world. They're a different reality, a social-cultural-historical reality. Mathematics is that third kind of reality—"inner" with respect to society at large, "outer" with respect to you or me individually.

To say mathematical objects are invented or created by humans makes them different from natural objects—rocks, X-rays, dinosaurs. Some philosophers (Stephen Körner, Hilary Putnam) argue that the subject matter of pure mathematics is the physical world—not its actualities, but its potentialities. "To exist in mathematics," they think, means "to exist potentially in the physical world." This interpretation is attractive, because it lets mathematics be meaningful. But it's unacceptable, because it tries to explain the clear by the obscure.

Consider this famous theorem of Georg Cantor: "If C is the set of points on the real line, and P is the set of all subsets of C, then it's impossible to put the points of C into 1–1 correspondence with the subsets of C—the elements of P." P can be regarded as the set of all functions of a real variable taking on the values 0 or 1. Nearly all these functions are nowhere continuous and nowhere measurable. We have no way to interpret them as physical possibilities.

The common sense of the working mathematician says this theorem is just a theorem of pure mathematics, not part of any physical interpretation. It's a human idea, recently invented. It wasn't timelessly or tenselessly existing, either as a Platonic idea or as a latent physical potentiality.

Why do these objects, our own creations, so often become useful in describing nature? To answer this in detail is a major task for the history of mathematics, and for a psychology of mathematical cognition that may be coming to birth in Piaget and Vygotsky. To answer it in general, however, is easy. Mathematics is part of human culture and history, which are rooted in our biological nature and our physical and biological surroundings. Our mathematical ideas in general match our world for the same reason that our lungs match earth's atmosphere. They were designed to match.

Mathematical objects can have well-determined properties because mathematical problems can have well-determined answers. To explain this requires investigation, not speculation. The rough outline is visible to anyone who studies or teaches mathematics. To acquire the idea of counting, we handle coins or beans or pebbles. To acquire the idea of an angle, we draw lines that cross. In higher grades, mental pictures or simple calculations are *reified* (term of Anna Sfard) and become concrete bases for higher concepts. These shared activities—first physical manipulations, then paper and pencil calculations—have a common product—shared concepts.

Not everyone achieves the desired result. The student who doesn't catch on doesn't pass the course.Why can we converse about polynomials? We've been trained to, by a training evolved for that purpose. We do it without a definition of "polynomial." Even without a definition, polynomial is a shared notion of middle-school students and teachers. And polynomials are objective: They have certain properties, whether we know them or not. These are implicit in our common notion, "polynomial."

To unravel in detail how we attain this common, objective notion is a deep problem, comparable to the problem of language acquisition. No one understands clearly how children acquire rules of English or Navajo, which they follow without being able to state them. These implicit rules don't grow spontaneously in the brain. They come from the shared language-use of the community of speakers. The properties of mathematical objects, like the properties of English sentences, are properties of shared ideas.

The observable reality of mathematics is this: an evolving network of shared ideas with objective properties. These properties may be ascertained by many kinds of reasoning and argument. These valid reasonings are called "proofs." They differ from one epoch to another, and from one branch of mathematics to another.

Looking at this fact of experience, we find questions. How are mathematical objects invented? What's the interplay of mathematics with the ideas and needs of science? How does proof become refined as errors are uncovered? Does the network of mathematical reasoning have an integrity stronger than any link, so that the fracture of any link affects only the closest parts?

These questions can be studied by historians of mathematics. Thomas Kuhn showed the insight that the history of science can give to the philosophy of science. Such work is beginning in the history and philosophy of mathematics.

Generally speaking, before an answer is interesting or even makes sense, there has to be a question. This trivial remark applies to mathematics as well as to anything else. Mathematical statements, mathematical theorems, are answers to questions. Modern mathematics has been sarcastically described as "answers to questions that nobody asked." This is unfair. Most likely the mathematician who found the answer did first ask the question. And very likely he'll publish the answer without mentioning the question. To an unwary reader it can then look like a self-subsisting, self-justifying piece of information, a question-less answer.

The mystery of how mathematics grows is in part caused by looking at mathematics as answers without questions. That mistake is made only by people who have had no contact with mathematical life. It's the questions that drive mathematics. Solving problems and making up new ones is the essence of mathematical life. If mathematics is conceived apart from mathematical life, of course it seems—dead.

To learn how mathematics grows, study how mathematical problems are recognized, how they're attractive. It has to be both something somebody would *like* to do and something somebody might be *able* to do.

An adequate description of today's mathematics (or any other period's) has to include some problems that are considered interesting. That's one reason a formal axiomatic description is incomplete and misleading.

This is recognized by Kitcher in his *Nature of Mathematical Knowledge*. It's implicit in Lakatos's *Proofs and Refutations*. It's fatally absent in Frege, Russell, and their epigones.

Psychological and historical studies won't make mathematical truth indubitable. But why expect mathematical truth to be indubitable? Correcting errors by confronting them with experience is the essence of science. What's needed is explication of what mathematicians do—as part of general human culture, as well as in mathematical terms. The result will be a description of mathematics that mathematicians recognize—the kind of truth that's obvious once said.

Certain kinds of ideas (concepts, notions, conceptions, and so forth) have science-like quality. They have the rigidity, the reproducibility, of physical science. They yield reproducible results, independent of particular investigators. Such kinds of ideas are important enough to have a name.

Study of the lawful, predictable parts of the physical world has a name: "physics." Study of the lawful, predictable, parts of the social-conceptual world also has a name: "mathematics."

defn. of math

A world of ideas exists, created by human beings, existing in their shared consciousness. These ideas have objective properties, in the same sense that material objects have objective properties. The construction of proof and counterexample is the method of discovering the properties of these ideas. This branch of knowledge is called mathematics.

An Objection

There's a logical difficulty we have to look at.

I say the 3-cube or the 4-cube—any mathematical object you like—exists at the social-cultural-historic level, in the shared consciousness of people (including retrievable stored consciousness in writing). In an oversimplified formulation, "mathematical objects are a kind of shared thought or idea."

A mathematical 3-cube is just an idea we share.

This statement is open to an objection. If you turn it around, as by ordinary logic it seems you have a right to do, you get "A certain idea we share is a mathematical 3-cube."

That is, an idea has volume, and vertices, edges, and faces—all of which is nonsense. Probe my mind-brain anyway you like; you won't find inside it a cube or a hyper-cube.

What are we trying to say?

Things become clear if we turn to familiar material objects. We have an idea of a chair, but our idea of a chair isn't a chair. It's our mind-brain's representation of a chair, analogous to a photograph of a chair or to the definition of "chair" in Webster. We know little about the construction or functioning of ideas in the mind-brain. But there's no logical confusion between a chair and the idea of a chair.

Between a 4-cube and the idea of such, there is a confusion. Why? Because we have nowhere to point, to show a "real" 4-cube as distinct from the idea of a 4-cube.

There are two ways to go from here. One well-worn path is the Platonist way. "There *is* a real 4-cube. It's a transcendental immaterial inhuman abstraction. Our idea of a cube is a representation of this transcendental thing, parallel to our idea of chair being a representation of real chairs."

The other way is fictionalism. There is no more a "real" 4-cube than a "real" Mickey Mouse. Oedipus and Mickey Mouse exemplify shared ideas that don't represent anything real. They show that there can be representation without a represented.

Our mental picture of a 4-cube is only a picture, not a 4-cube. It doesn't have vertices or edges, but it does have representations of vertices and edges. It's different from a 4-cube, because it does exist (on the social-cultural-historic level) while the 4-cube, itself doesn't exist. Or, as I prefer to say, it exists only in its social and mental representations.

A 4-cube has 16 vertices. At each vertex, 4 edges meet at right angles. But there is no 4-cube! So *nothing* has 16 vertices at which 4 edges meet at right angles—except as we have a shared idea of such a thing, an idea so consistent, rigid, and reliable that we share each other's reasonings, and come to the same conclusions.

This may sound paradoxical. It's an honest account of the actual state of affairs.

No purpose
or ultimate cause.

It's a Futile Question

Some questions, which at first seem meaningful, are *futile*—to answer them is neither possible nor necessary.

Why are there rigid, reproducible concepts such as number or circle?

Why is there consciousness?

Why is there a cosmos?

We need not answer Kant's question, "How is mathematics possible?" any more than we need answer Heidegger's question, "Why should anything exist?"

I haven't heard about progress on either problem.

People who think up such questions may get compliments for asking amusing questions. But no physicist and few philosophers feel obliged to answer Heidegger's question. The existence of a world is the starting point from which we go forward.

Once upon a time an important question was, "How can the world be so simple, complicated, and beautiful unless Someone made it?" Now many would say that's a futile question.

Some of today's questions about cosmology, ethics, determinism, or cognition may be futile.

Kant answered his question, "How is mathematics possible?" If not because of the existence of external mathematical objects, then, he thought, our minds ("intuitions") must impose arithmetic and geometry universally.

Ethnology, comparative history, developmental psychology, the development of non-Euclidean geometry, and general relativity, all show that Euclidean geometry is not built into everyone's mind/brain. We think about space in more than one way. We reject Kant's answer. Must we still accept his question?

I counter Kant's question with a counter-question: "Why should your question have an answer?"

validation of experience

This much is clear: Mathematics *is* possible. It's the old saying, "What *is* happening *can* happen."

How does mathematics come about, in a daily, down-to-earth sense? That question belongs to psychology, to the history of thought, and to other disciplines of empirical science. It can't be answered by philosophy. Accept the *possibility* of mathematics as a fact of experience.

Major empirical discoveries about it are coming. Neuro-scientists are hunting for the brain structures we use in counting and spatial thinking. George Lakoff, George Johnson, Terry Regier, and others, using work of Antonio Damasio, Gerald Edelman, and others, may be approaching that goal.

When such discoveries come they'll have tremendous importance, both scientific and practical. But they won't decide philosophical controversies.

To see why not, consider a comparable question. Is what our eyes see really there? That is, is matter an illusion, as many brilliant idealists have said? Or, as Kant taught, is it impossible for us to know whether it's an illusion?

These questions have been of the highest concern to great philosophers.

Today, we realize that those philosophers had limited understanding of the workings of the eye and brain. We do know something about those workings. Maybe some day we'll understand them completely, for practical purposes. Would that understanding tell us whether the visible is real? No. Idealists and skeptics could find new distinctions, and go on being idealists or skeptics as long as they wished.

The reasons that apply to visual reality apply to mathematical reality. The philosophical issues around it will be influenced by empirical discovery, but not settled.

We can study how mathematics develops, in history, in society, and in the individual. We can study how mathematical theories give rise to one another. We can study how mathematics springs from and goes back to physics and other sciences. But the question, "How is mathematics possible?" tries to push mathematics into a pigeonhole: physical, mental, transcendental. None fits. I reject the question and its old alternatives.

Since Dedekind and Frege in the 1870s and 1880s, philosophy of mathematics has been stuck on a single problem—find a solid foundation to which all mathematics can be reduced, a *foundation* to make mathematics indubitable, free of uncertainty, free of any possible contradiction (see section in Chapter 8).

That goal is now admitted to be unattainable. Yet, with the exception of a few mavericks, philosophers continue to see "foundation" as the main interesting problem in philosophy of mathematics.

The key assumption in all three foundationist viewpoints is mathematics as a source of indubitable truth. Yet daily experience finds mathematical truth to be fallible and corrigible, like other kinds of truth.

None of the three can account for the existence of its rivals. If Platonism is right, the existence of formalism and constructivism is incomprehensible. If constructivism is right, the existence of Platonism and formalism is incomprehensible. If formalism is right, the existence of Platonism and constructivism is incomprehensible.

Humanism sees that constructivism, formalism, and Platonism each fetishizes one aspect of mathematics, insists that one limited aspect *is* mathematics.

This account of mathematics looks at what mathematicians do. The novelty is conscious effort to avoid falsifying or idealizing.

If we give up the obligation of mathematics to be a source of indubitable truths, we can accept it as a human activity. We give up age-old hopes, but gain a clearer idea of what we are doing, and why.

Humanism.

1. Mathematics is human. It's part of and fits into human culture.
2. Mathematical knowledge isn't infallible. Like science, mathematics can advance by making mistakes, correcting and recorrecting them. (This fallibilism is brilliantly argued in Lakatos's Proofs and Refutations.)
3. There are different versions of proof or rigor, depending on time, place, and other things. The use of computers in proofs is a nontraditional rigor. Empirical evidence, numerical experimentation, probabilistic proof all help us decide what to believe in mathematics. Aristotelian logic isn't always the only way to decide.
4. Mathematical objects are a distinct variety of social-historic objects. They're a special part of culture. Literature, religion, and banking are also special parts of culture. Each is radically different from the others.

Music is an instructive example. It isn't a biological or physical entity. Yet it can't exist apart from some biological or physical realization—a tune in your head, a page of sheet music, a high C produced by a soprano, a recording, or a radio broadcast. Music exists by some biological or physical manifestation, but it makes sense only as a mental and cultural entity.

What confusion would exist if philosophers could conceive only two possibilities for music—either a thought in the mind of an Ideal Musician, or a noise like the roar of a vacuum cleaner.

I have two concluding points.

Point 1 is that mathematics is a social-historic reality. This is not controversial. All that Platonists, formalists, intuitionists, and others can say against it is that it's irrelevant to their concept of philosophy.

Point 2 *is* controversial: There's no need to look for a hidden meaning or definition of mathematics beyond its social-historic-cultural meaning. Social-historic is all it needs to be. Forget foundations, forget immaterial, inhuman "reality."

There is no purpose