Philosophy of Mathematics
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~~that are constructive in this sense not because he accepted the philosophy of intuitionism, but because those constructive methods, being more limited in scope, are markedly less doubtful as regards their consistency than is the nonconstructive reasoning of systems such as *Principia Mathematica*. Hilbert felt that all mathematicians, including intuitionists, could thus agree about the validity of meta-mathematical reasoning. Intuitionists might regard the formalized systems under study as incapable of being interpreted so as to come out true; yet nevertheless they should accept the meta-theorems that were proved about these systems.~~

Incompletability

In addition to the question of consistency, another important question to ask about deductive systems is the question of their completeness. The idea of completeness is the idea that nothing which ought to be a theorem of the system fails to be provable as a theorem. An interpreted system of geometry, for example, would be said to be complete (deductively complete) provided that every true statement expressible in its primitive terms is a theorem. In talking about the completeness of an uninterpreted system we would not want to discuss truth, so we reformulate the idea of completeness thus: to say that an uninterpreted system is complete is to say that there is no sentence expressible in the primitive terms of the system such that neither it nor its denial is provable as a theorem of the system. Analogously, a formalized system is said to be complete provided that each well-formed formula is a theorem either when it is written without or when it is written with the mark "—" prefixed (again assuming that this mark is one that occurs in the system and one that would normally be interpreted as expressing denial). Completeness is a desirable property for a system to have, of course. If an interpreted system is incomplete, this means that there are truths about its subject matter which cannot be deduced from its axioms; its axioms fail to capture all the information that we would have liked them to contain.

Hilbert's meta-mathematical method is appropriate for investigating completeness as well as consistency. Hilbert, and other mathematicians and logicians of the earlier part of this century, confidently hoped that it would eventually prove possible to develop each branch of mathematics in the form of an axiomatic system that could be shown to be both consistent and complete; or, better still, to develop one unified system for the whole of mathematics, which could be shown to be both consistent and complete. This attractive and plausible expectation was decisively destroyed by the work of Gödel in 1931. By an ingenious chain of meta-mathematical reasoning, Gödel was able to demonstrate that for systems of the most important kind, consistency is incompatible with completeness. Such systems, if consistent, must necessarily be incomplete.

The kind of systems which Gödel studied were those such as *Principia Mathematica*, whose primitive terms and axioms are rich enough to allow one to speak of the natural numbers and to deduce the laws governing such operations as addition and multiplication of them. The formulas of such a system, when interpreted in the intended way, speak of natural numbers; nevertheless, by means of a technique now called Gödel numbering, he was able to show how some formulas of the system must necessarily *reflect* meta-mathematical assertions about the system itself. That is, he was able to find a way of correlating certain formulas of the system with meta-mathematical statements about the system so that each such formula under its normal interpretation expresses a true statement about natural numbers just in case the meta-mathematical assertion with which it is correlated is true also. Gödel's proof

In developing these correlations, Gödel started with the primitive signs of the system; for each primitive sign of the system a particular natural number is arbitrarily chosen and is defined as being its Gödel number. A definition is then offered of what numerical properties a number shall have in order to be called the Gödel number of a formula; and this is done in such a way that the Gödel number of any formula is a numerical function of the Gödel numbers of the primitive signs that occur in it and of their order of occurrence, so that given any Gödel number one can calculate what formula it belongs to. What it is for a number to be the Gödel number of a sequence of formulas can then be defined. Ultimately it is possible to define what numerical properties a number must have in order to be called the Gödel number of a proof; that is, the Gödel number of a sequence of formulas, each of which either is an axiom or is obtained from earlier formulas in the sequence by means of a transformation rule of the system. Then it is possible to define the numerical properties that a number must have in order to be called the Gödel number of a theorem (that is, the last formula in a sequence that is a proof); and thence to define the numerical properties that a number must have if it is to be called the Gödel number of a well-formed formula that is not a theorem.

These definitions are carefully constructed so that the assertions about Gödel numbers will be true just in case the meta-mathematical assertions associated with them are true. Thus, the assertion that a certain number has the numerical properties that entitle it to be called the Gödel number of a non-theorem will be true just in case the rules of the system do not make it possible to construct a proof of the formula whose Gödel number this number is. The assertions about

Gödel numbers are in turn correlated with formulas of the system: each assertion is correlated with the formula which, when its signs are interpreted in the normal way, will express that assertion. Thus, certain formulas of the system are correlated with assertions about Gödel numbers, and those in turn are correlated with meta-mathematical assertions, giving us a correlation between these formulas of the system and meta-mathematical assertions.

Now let us consider a formula which is correlated with the assertion that a specific natural number has those numerical properties that make it the Gödel number of a well-formed formula which is not a theorem. Since it can be interpreted as making a statement about a particular number, this formula has to contain a numeral (a sign, or sequence of signs, which under its normal interpretation would be understood as naming a particular number). The formula itself has a Gödel number, and to that Gödel number there corresponds some numeral. Now suppose that the numeral in this formula is the very numeral which corresponds to the Gödel number of the formula itself. Gödel showed how to write out a formula like this. Because of the way the formulas of the system reflect meta-mathematics, this special Gödelian formula (when normally interpreted) expresses a true statement about natural numbers just in case the meta-mathematical assertion with which it is correlated is true. That assertion is the assertion that this very formula is not a theorem. Thus we have a formula which is not a theorem if it expresses a truth about the natural numbers, and is a theorem if it expresses a falsehood about the natural numbers.

By further reasoning it is possible to demonstrate that the existence of this formula means that the system must be outright inconsistent if it is complete. It can be consistent only at the price of being incomplete, and can be complete only at the price of being inconsistent. In this sense, the system is said to be incompletable. Gödel's reasoning showed that this conclusion applies to any system rich enough to express the theory of the natural numbers; for in any such system some Gödelian formula can be constructed.

One way of formulating this conclusion is to say that any consistent axiomatization of the theory of the natural numbers must always fail to capture as theorems all the truths about the natural numbers. Some axiomatizations can capture more of the truths about natural numbers than others do, and for each truth there is some axiomatization in which it is captured; but no single consistent axiomatization can get them all. This result strikes a decisive blow against the idea that mathematical truth can be identified with deducibility from axioms.

A further conclusion which Gödel established as a consequence of this is that any meta-mathematical demonstration of the consistency of systems of this kind must employ meta-mathematical principles more complex than the principles embodied within the system under study. The consistency of a system such as *Principia Mathematica* cannot be demonstrated except in a meta-language that uses reasoning logically richer and more complex than that of *Principia Mathematica* itself. This result dashed Hilbert's hope of using generally accepted meta-mathematical methods to establish the consistency of mathematical systems; for it means that there are many systems such that if any meta-mathematical arguments can be given in favor of their consistency, these can only be arguments employing nonconstructive methods not acceptable to all mathematicians. Moreover, these methods must often be more complex and therefore more suspect as regards their consistency, than is the system under study.

Formalism

In the preceding chapter we considered literalistic views of the mathematics of number, views which hold that this mathematics possesses its intellectual value because its laws can be interpreted as important truths. Among these views, nominalism seemed incapable of providing any true interpretation, and conceptualism seemed to rest upon a cloudy and unsubstantiated doctrine regarding the mind's supposed creative powers. Realism seemed the least unpromising literalistic view. Yet realism does not stand up well in the light of subsequent developments. If the mathematics of number is the investigation of a field of independently real abstract entities such as sets and numbers, then there ought to be some one true body of laws about these entities. Yet in trying to meet the challenge of the paradoxes, mathematical logicians have developed four basically different kinds of theories, whose laws are by no means entirely in agreement. Each kind of theory is somewhat arbitrary and makeshift, and there seems to be no ground whatever for holding that one of these approaches is truer than the others. The presence of different, partly conflicting theories and the absence of any ground for calling one truer than the others make the realistic philosophy much less tenable than it seemed at first. We cannot feel that we are discovering truths about an independent reality, under these circumstances. Furthermore, Gödel's demonstration of the incompletability of number theory is another blow to the realistic philosophy. If it were an independent reality of sets and numbers that mathematics describes, then one would have expected that the truth about that reality, which would have to be consistent, should allow of being axiomatized completely. A reality the truth about which is necessarily incapable of being described in any complete manner seems a queer and suspect sort of reality—that is, not a reality at all.