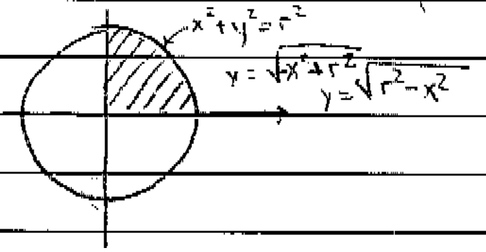
Notetaker's name ______ Class ______ Date May 8.09

Section 7.3

Trig Substitution.

Ex:1 Derive the area of a circle with radius r

(the answer : $A = \pi r^2$)



$A = 4\int_0^r \sqrt{r^2-x^2}\ dx.$

try a trig substitution

$x = r\sin\theta$

$dx = r\cos\theta\ d\theta$

indef. int. $\int \sqrt{r^2-x^2}\ dx = \int \sqrt{r^2 - r^2 \sin^2\theta} \cdot r\cos\theta\ d\theta$

$= \int r\sqrt{1-\sin^2\theta} \cdot r\cos\theta\ d\theta = r^2 \int \sqrt{\cos^2\theta}\ \cos\theta\ d\theta = r^2 \int |\cos\theta|\ \cos\theta\ d\theta$

when note $\sqrt{x^2} = |x|$

required $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

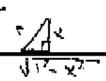
$r^2 \int \cos^2\theta\ d\theta = \frac{r^2}{2}\int 1+\cos 2\theta\ d\theta$

$\Rightarrow \frac{r^2}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right] + C$

recall : $x = r\sin\theta$

$\theta = \arcsin\left(\frac{x}{r}\right)$

$\frac{1}{2}\sin 2\theta = \underbrace{\sin\theta}_{\frac{x}{r}}\ \underbrace{\cos\theta}$



$\cos\theta = \frac{\sqrt{r^2-x^2}}{r}$

$\Rightarrow \frac{r^2}{2}\left[\sin^{-1}\left(\frac{x}{r}\right) + \frac{x\sqrt{r^2-x^2}}{r^2}\right] + C.$

$A = \int_0^r 4\sqrt{r^2-x^2}\ dx = \frac{4r^2}{2}\left[\sin^{-1}\left(\frac{x}{r}\right) + \frac{x\sqrt{r^2-x^2}}{r^2}\right]_0^r = 2r^2\left(\underbrace{\sin^{-1}(1)}_{\frac{\pi}{2}} - \underbrace{\sin^{-1}(0)}_{0}\right) =$

$= 2r^2 \cdot \frac{\pi}{2} = \pi r^2.$

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Trig Substitution.

$a^2 - x^2 \longrightarrow x = a \sin\theta$

$a^2 + x^2 \longrightarrow x = a \tan\theta$

$x^2 - a^2 \longrightarrow x = a \sec\theta$.

Ex: 2: $\int x^3 \sqrt{x^2+4}\, dx$

Trig subs:

$x = 2\tan\theta$

$dx = 2\sec^2\theta \, d\theta$.

$= \int 8\tan^3\theta \sqrt{4\tan^2\theta + 4} \cdot 2\sec^2\theta \, d\theta$.

$= 32 \int \tan^3\theta \cdot \sec^2\theta \sqrt{\underbrace{\tan^2\theta + 1}_{\sec^2\theta}} \, d\theta$

$= 32 \int \tan^3\theta \cdot \sec^2\theta \, |\sec\theta| \, d\theta$ $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$.

$= 32 \int \tan^3\theta \cdot \sec^3\theta \, d\theta$

$= 32 \int (\sec^2\theta - 1) \sec^3\theta \cdot \tan\theta \, d\theta$ let $u = \sec\theta$

$= 32 \int (u^2 - 1) u^2 \, du$ $du = \sec\theta \tan\theta \, d\theta$

$= 32 \int u^4 - u^2 \, du = 32 \left[\frac{u^5}{5} - \frac{u^3}{3} \right] + C$

$= 32 \left[\frac{1}{5}\sec^5\theta - \frac{1}{3}\sec^3\theta \right] + C$.

we know $\frac{x}{2} = \tan\theta$

$= \frac{32}{5}\left(\frac{\sqrt{x^2+4}}{2}\right)^5 - \frac{32}{3}\left(\frac{\sqrt{x^2+4}}{2}\right)^3 + C$

$\sec\theta = \frac{\sqrt{x^2+4}}{2}$

$= \frac{1}{5}(x^2+4)^{5/2} - \frac{4}{3}(x^2+4)^{3/2} + C$.

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Section 7.3

Trig. Substitution

$a^2 - x^2 \Rightarrow x = a\sin\theta$

$a^2 + x^2 \Rightarrow x = a\tan\theta$

$x^2 - a^2 \Rightarrow x = a\sec\theta$

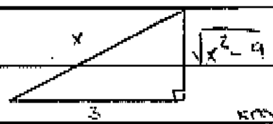
Ex: 3 $\int \frac{\sqrt{x^2-9}}{x^4}\,dx \Rightarrow \int \frac{\sqrt{9\sec^2\theta-9}}{81\sec^4\theta}\cdot 3\sec\theta\tan\theta\,d\theta \Rightarrow$ let $x = 3\sec\theta$

$\int \frac{3\sqrt{\sec^2\theta-1}}{81\sec^4\theta}\cdot 3\sec\theta\tan\theta\,d\theta \Rightarrow \int \frac{\tan^2\theta}{9\sec^3\theta}\,d\theta$ $dx = 3\sec\theta\tan\theta\,d\theta$

$\Rightarrow \frac{1}{9}\int \frac{\sin^2\theta}{\cos^2\theta}\cdot\cos^3\theta\,d\theta \Rightarrow \frac{1}{9}\int \sin^2\theta\cdot\cos\theta\,d\theta$

$\Rightarrow \frac{1}{9}\int u^2\,du \Rightarrow \frac{1}{9}\cdot\frac{u^3}{3} \Rightarrow \frac{1}{27}u^3 + C.$ let $u = \sin\theta$

$\Rightarrow \frac{1}{27}\sin^3\theta + C \Rightarrow \frac{1}{27}\frac{(x^2-9)^{3/2}}{x^3} + C.$ $du = \cos\theta\,d\theta$



know $x = 3\sec\theta$

$\frac{x}{3} = \sec\theta$

Ex: 4 $\int_0^{2/3} x^3\sqrt{4-9x^2}\,dx \Rightarrow 3\int_0^{2/3} x^3\sqrt{\frac{4}{9}-x^2}\,dx$ let $x = \frac{2}{3}\sin\theta$

$dx = \frac{2}{3}\cos\theta\,d\theta$

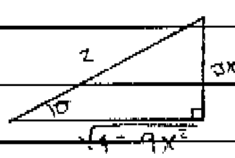
Consider $3\int x^3\sqrt{\frac{4}{9}-x^2}\,dx \Rightarrow 3\int \frac{8}{27}\sin^3\theta\sqrt{\frac{4}{9}-\frac{4}{9}\sin^2\theta}\cdot\frac{2}{3}\cos\theta\,d\theta$

$\Rightarrow 3\cdot\frac{8}{27}\cdot\frac{2}{3}\cdot\frac{2}{3}\int \sin^3\theta\cos^3\theta\,d\theta \Rightarrow \frac{32}{81}\int \sin^3\theta\cos^2\theta\,d\theta$

$\Rightarrow \frac{32}{81}\int (1-\cos^2\theta)\cos^2\theta\cdot\sin\theta\,d\theta \Rightarrow$ let $u = \cos\theta$

$\Rightarrow -\frac{32}{81}\int u^2 - u^4\,du \Rightarrow -\frac{32}{81}\left(\frac{u^3}{3}-\frac{u^5}{5}\right) + C.$ $du = -\sin\theta\,d\theta$

$\Rightarrow -\frac{32}{81}\left(\frac{\cos^3\theta}{3}-\frac{\cos^5\theta}{5}\right) + C.$ $-du = \sin\theta\,d\theta$



$x = \frac{2}{3}\sin\theta$

$\sin\theta = \frac{3x}{2}$

$\Rightarrow -\frac{32}{81}\left(\frac{(4-9x^2)^{3/2}}{24}-\frac{(4-9x^2)^{5/2}}{32\cdot 5}\right) + C.$

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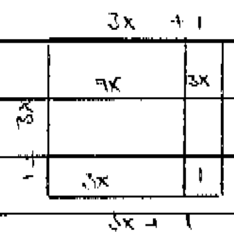
Evaluate the definite integral:

$$= \left[\frac{-4}{243}\left(4-9x^2\right)^{3/2} + \frac{1}{405}\left(4-9x^2\right)^{5/2} \right]_0^{2/3}$$

$$= (0+0) - \left(-\frac{32}{243} + \frac{32}{405}\right) = \frac{64}{1215}$$

Ex:5 $\displaystyle\int \frac{dx}{\sqrt{9x^2+6x-8}}$

complete the square



$$9x^2+6x-8 = (9x^2+6x+1)-8-1$$

$$= (3x+1)^2-9$$

$$\int \frac{dx}{\sqrt{(3x+1)^2}} = \frac{1}{3}\int \frac{du}{\sqrt{u^2-9}} \Rightarrow \frac{1}{3}\int \frac{3\sec\theta\tan\theta\, d\theta}{\sqrt{9\sec^2\theta-9}}$$

subs. let $u = 3x+1$

$$\Rightarrow \frac{3}{9}\int \frac{\sec\theta\tan\theta\, d\theta}{\tan\theta} \Rightarrow \frac{1}{3}\int \sec\theta\, d\theta$$

$du = 3\,dx$

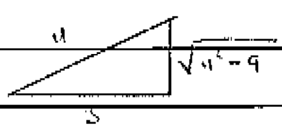
$$\Rightarrow \frac{1}{3}\ln\left|\sec\theta + \tan\theta\right| + C$$

$\frac{1}{3}du = dx$

$\frac{u}{3} = \sec\theta$

trig. let $u = 3\sec\theta$

$du = 3\sec\theta\tan\theta\, d\theta$



$$\Rightarrow \frac{1}{3}\ln\left|\frac{u}{3} + \frac{\sqrt{u^2-9}}{3}\right| + C$$

$$\Rightarrow \frac{1}{3}\ln\left|\frac{3x+1}{3} + \frac{\sqrt{(3x+1)^2-9}}{3}\right| + C.$$