5.3 : Fundamental Thm. of Calculus

Fundamental means important.

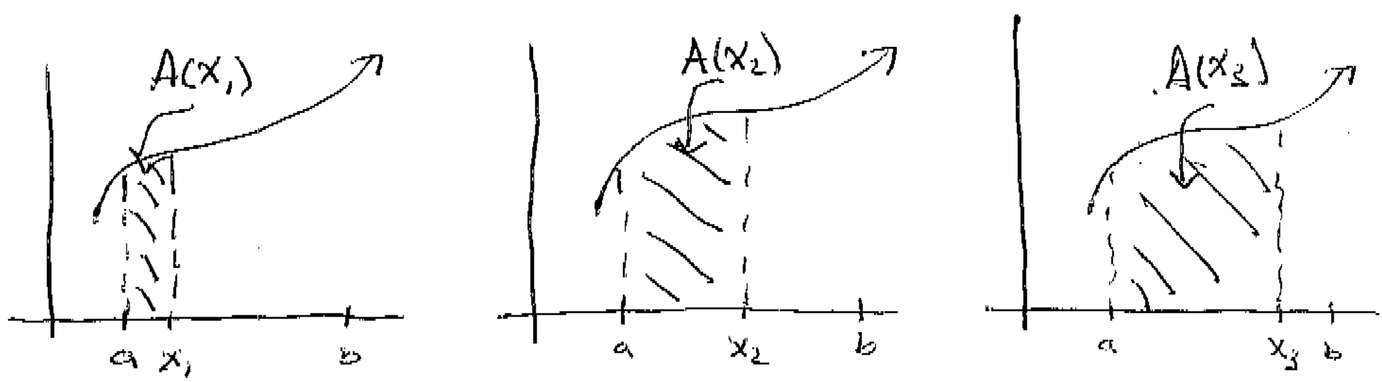
Recall the definition of the definite integral

$$\underbrace{\int_a^b f(x)\,dx}_{\text{a number}} = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

our goal is to functionize the definite integral.

We do this by developing a cumulative area under the curve fct.

$$A(x) = \int_a^x f(t)\,dt$$ is the cumulative area fct. and "t" is a dummy variable



Ex 1: (next pages)

Mathematica notebook.

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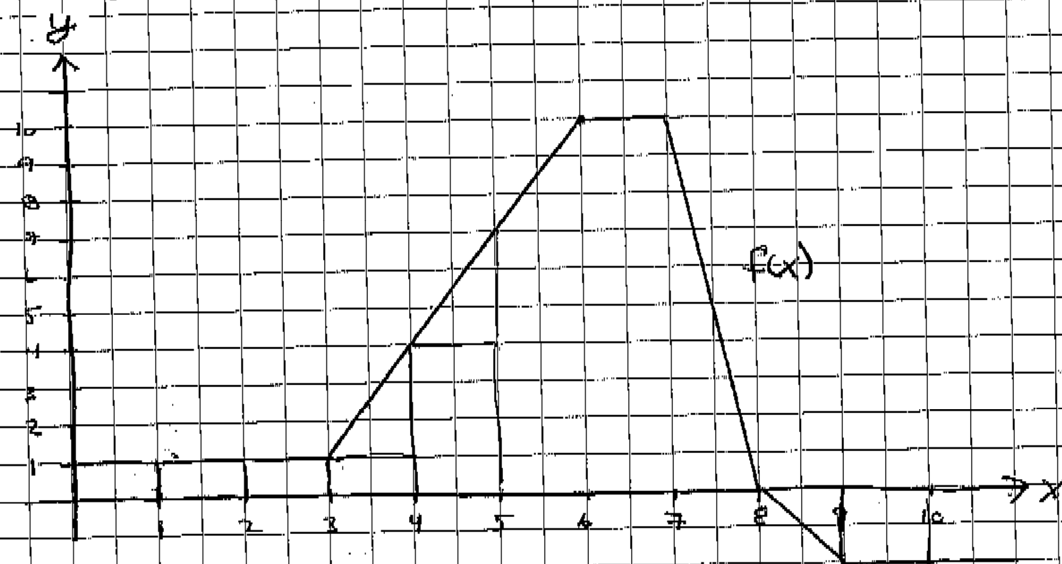
Ex1: Sketch $A(x) = \int_1^x f(t)\,dt$ for the given graph of $f(x)$.



$A(1) = \int_1^1 f(t)\,dt = 0$ $A(5) = 10$ $A(9) = 32.5$

$A(2) = \int_1^2 f(t)\,dt = 1$ $A(6) = 18.5$ $A(10) = 30.5$

$A(3) = \int_1^3 f(t)\,dt = 2$ $A(7) = 28.5$

$A(4) = \int_1^4 f(t)\,dt = 4.5$ $A(8) = 33.5$

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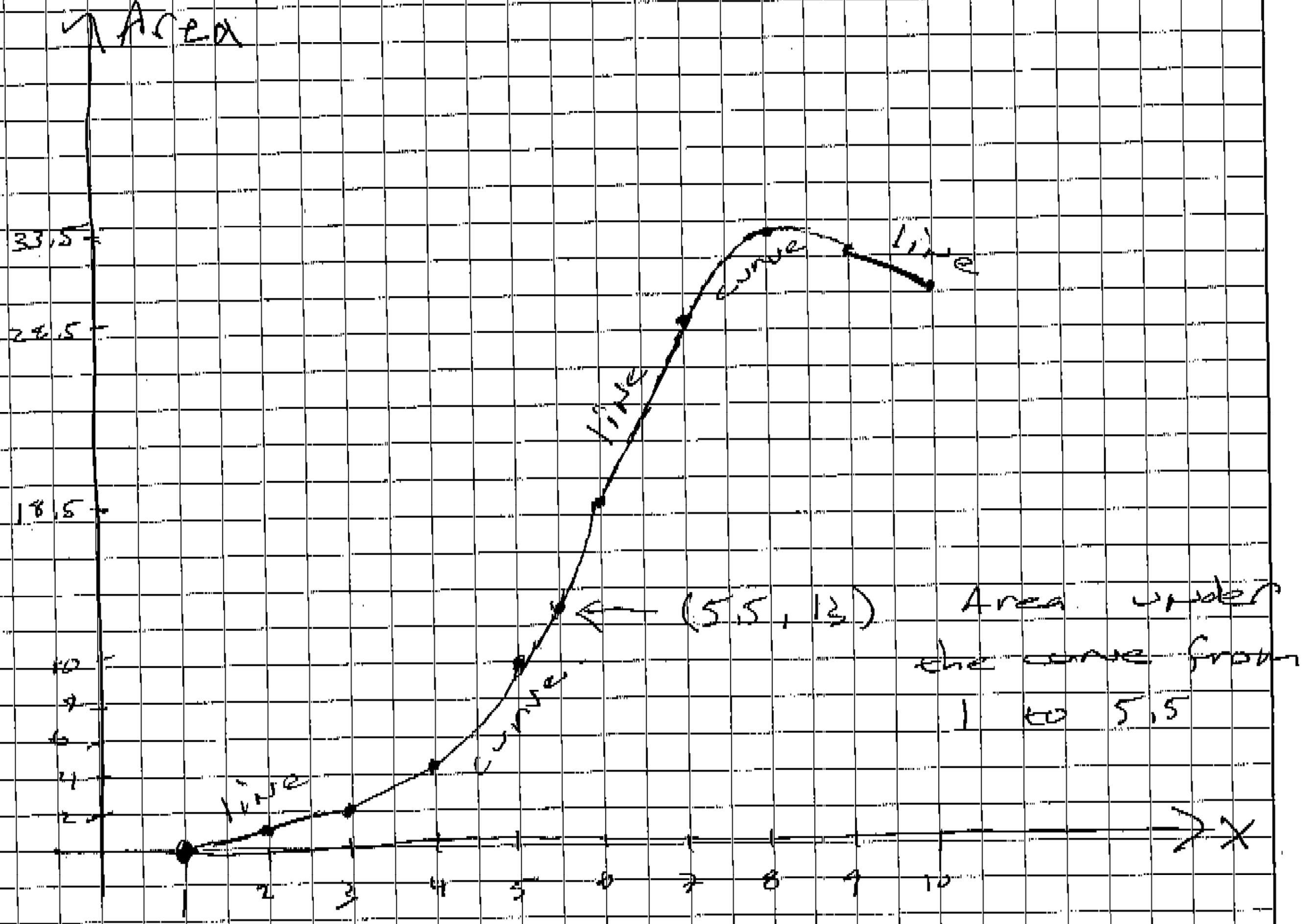
t	1	2	3	4	5	6	7	8	9	10
A	0	1	2	4.5	10	18.5	28.5	33.5	32.5	30.5



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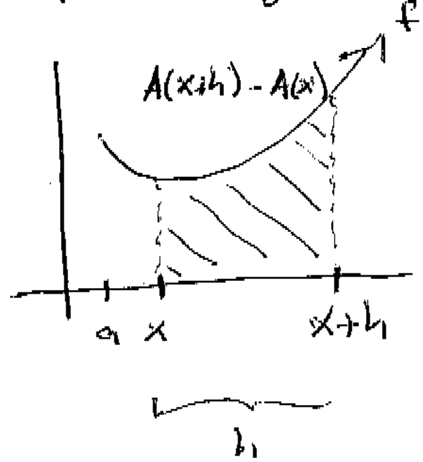
Derivation of FToC in three Steps

Step 1: show $\frac{dA}{dx} = f(x)$.

This says, "RoC of A WRT x is the length of the right side of the region."

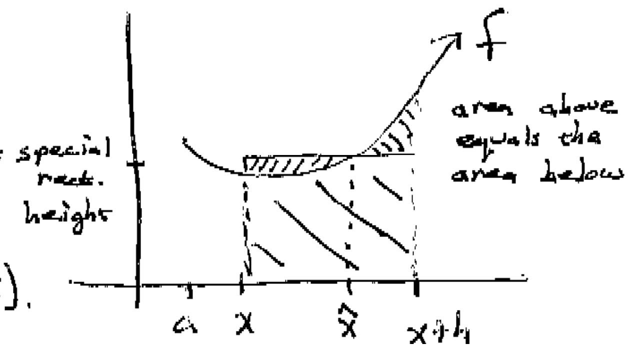


$$\frac{dA}{dx} = \lim_{h \to 0} \frac{A(x+h) - A(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(\hat{x})\, h}{h}$$

$$= \lim_{h \to 0} f(\hat{x})$$

$= f(x)$. since f is continuous.



$x \leq \hat{x} \leq x+h$

according to the 1st MVT of Integral Calculus This $\hat{x}$ exists giving that special rect. height.

Step 2: Find a formula for $A(x)$.

$A(x)$ is an antideriv. of $f(x)$.

But, if $F(x)$ is any antideriv. of f, we know $A(x) = F(x) + c$ for some constant c.

Let's find c:

$A(a) = 0$

$\Rightarrow 0 = F(a) + c$

$\Rightarrow c = -F(a)$

$\Rightarrow A(x) = F(x) - F(a)$.

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Step 3: Conclusion.

since $A(x) = F(x) - F(a)$

and

$$A(x) = \int_a^x f(t)\,dt,$$

we conclude that $\int_a^b f(x)\,dx = F(b) - F(a)$

Fundamental Thm. of Calculus:

If f is continuous on $[a, b]$ and if F is any antiderivative of f, then $\int_a^b f(x)\,dx = F(b) - F(a)$.

ex 2: $\int_2^4 x^3\,dx$

ex 3: $\int_1^{e^3} \frac{dx}{x}$

ex 4: $\int_{-2}^{2} \frac{dx}{x^3}$ <- use a fourth power to make the point of this example more obvious to the students.

ex 5: $\int_0^{\pi} \sin(x)\,dx$

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ex 6: $\frac{d}{dx}\int_0^x \cos(\sin(\ln(t)))\,dt$ <- better begin at t=1 so as to maintain continuity

If $A(x) = \int_0^x \cos(\sin(\ln(t)))\,dt$, then we want $\frac{dA}{dt}$. But, we already showed that this is $f(x) = \cos(\sin(\ln(t)))$.

ex 7: $\frac{d}{dx}\int_1^{x^2} \frac{\cos(t)}{t}\,dt$.

If $A(x) = \int_1^x \frac{\cos(t)}{t}\,dt$, then we want

$$\frac{d}{dx}A(x^2) = \cancel{A}\, f(x^2)\cdot 2x$$

$$= \frac{\cos(x^2)}{x^2}\cdot 2x$$

an example of a fct such as this is the error fct. from statistics:

$$\mathrm{Erf}(x) = \int_0^x \frac{2}{\sqrt{\pi}} e^{-t^2}\,dt$$

ex 8: If $g(x) = \int_{\sin x}^{0} (t + \cos(t))\,dt$, find $g'(x)$.

If $A(x) = \int_0^x (t + \cos(t))\,dt$, then we want

$$\frac{d}{dx} -g(\sin(x)) = -f(\sin(x))\cdot\cos(x)$$

$$= -(\sin x + \cos(\sin x))\cos x$$