Course: Math 125 Tan Pham

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Honor Problems

Chapter 5, Problem Plus, page 413

1. If $x \sin(\pi x) = \int_0^{x^2} f(t)\, dt$, where f is a continuous function, find f(4).

<u>Solve:</u>

One approach to solve this problem is to use the Fundamental Theorem of Calculus Part I* (FToCp1). Since this is not in the usual form of the integral $\int_a^x f(x)dx$, we cannot apply the theorem. However, what we can do is to transform the integral itself.

So, letting $g(x) = \int_0^x f(t)dt$, by the FToCp1, we have

$$g'(x) = f(x)$$

And $g(x^2) = \int_0^{x^2} f(t)dt$.

Our target is to solve for f(4). In order to do that, we have to know what is f(x), or $g'(x)$.

Let's make an investigation.

Our only clue is $g'(x) = f(x)$. So, how do we find the derivative of g(x)? The only "relative" of g(x) here is $g(x^2)$, so let's see what is the derivative of $g(x^2)$, since we are looking for the derivative of g(x).

By the chain rule,

$$\frac{d}{dx} g(x^2) = 2x\,(g'(x^2))$$

Amazing, Amazing, Amazing!

Did you see the connection? Here we have f(x) in the disguise of $g'(x)$. The derivative of $g(x^2)$ is $2x\,(g'(x^2))$. SO,

If

$$g'(x) = f(x)$$

Then

$$g'(x^2) = f(x^2)$$

And do you know what else? The derivative of $g(x^2)$ is the derivative of $xsin(\pi x)$. (Re-look at the original question.) In another words, if you differentiate both sides of the expression, you will come up to the same result. Check for yourself.

Anyway, if we can find the derivative of $xsin(\pi x)$, then we will be done. How? Am I too fast? Let us gather all the clues one more time. We know that

$$g'(x) = f(x), \text{ so } g'(x^2) = f(x^2)$$

And

$$\frac{d}{dx}g(x^2) = 2x\,(g'(x^2))$$

Also,

$$\frac{d}{dx}xsin(\pi x) = \frac{d}{dx}\int_0^{x^2} f(t)dt = \frac{d}{dx}\,g(x^2)$$

What else? By the chain rule,

$$\frac{d}{dx}xsin(\pi x) = \sin(\pi x) + x\pi\cos(\pi x)$$

Thus,

$$2x\,f(x^2) = \sin(\pi x) + x\pi\cos(\pi x)$$

Or,

$$f(x^2) = \frac{\sin(\pi x) + x\pi\cos(\pi x)}{2x}$$

This is not f(x); nevertheless, it works.

Remember that our target is f(4). So, if x^2 = 4, then x must be 2**. Therefore, our last technique to use in this problem is substitution.

Substitute x = 2 into $f(x^2)$, we get

$$\boxed{f(4) = \frac{\sin(2\pi) + 2\pi\cos(2\pi)}{2(2)} = \frac{2\pi}{4} = \frac{\pi}{2}}$$

Hence, I conclude that our target, namely f(4), is $\frac{\pi}{2}$. My work here is done.

** Actually x can be negative 2 also. But it wouldn't change the result. Check for yourself.

* Check Glossary in the end for the Fundamental Theorem of Calculus, Part I.

Ready for some more? Here is another case, only more challenging, nevertheless, more interesting.

5. If $f(x) = \int_0^{g(x)} \frac{1}{\sqrt{1+t^3}} dt$, where $g(x) = \int_0^{\cos(x)} [1+\sin(t^2)] dt$, find $f'(\frac{\pi}{2})$.

We will use the same approach: the Fundamental Theorem of Calculus, Part 1.

By the look of this scenario, I say we should divide the task into two small sections.

Section 1: Find the derivative of $f(x) = \int_0^{g(x)} \frac{1}{\sqrt{1+t^3}} dt$

Let $m(x) = \int_0^x \frac{1}{\sqrt{1+t^3}} dt$, thus $m'(x) = \frac{1}{\sqrt{1+x^3}}$

So,

$$m(g(x)) = \int_0^{g(x)} \frac{1}{\sqrt{1+t^3}} dt = f(x)$$

Our target is $f'(\frac{\pi}{2})$. So we should look for something $f'(x)$-like.

This should be obvious,

$$f'(x) = \frac{d}{dx} m(g(x)) = g'(x)(m'[g(x)])$$

And since $m'(x) = \frac{1}{\sqrt{1+x^3}}$, $(m'[g(x)]) = \frac{1}{\sqrt{1+[g(x)]^3}}$, we have

$$f'(x) = g'(x)(m'[g(x)]) = g'(x)\frac{1}{\sqrt{1+[g(x)]^3}} \quad (1)$$

Now, here is the problem. What is the derivative of g(x)?

Don't panic! We still have another section to investigate.

Section 2: $g(x) = \int_0^{\cos(x)} [1+\sin(t^2)] dt$

Let $h(x) = \int_0^x [1+\sin(t^2)] dt$, thus $h'(x) = 1 + \sin(x^2)$.

So,

$$h(cosx) = \int_0^{cosx} [1 + \sin(t^2)]dt = g(x)$$

Now, what we want is g’(x), thus,

$$g'(x) = h'(\cos(x)) = -(\sin(x))(h'[\cos(x)])$$

Since $h'(x) = 1 + \sin(x^2)$, $h'(cosx) = 1 + \sin([\cos(x)]^2)$.

So,

$$g'(x) = -(\sin(x))(1 + \sin([\cos(\mathrm{x})]^2))$$

That is a relief. Now that we have g’(x), we can substitute it back into @, and find f’(x).

From (1),

$$f'(x) = g'(x)\frac{1}{\sqrt{1 + [g(x)]^3}}$$

Now, let me remind you that the problem asks us to find $f'\left(\frac{\pi}{2}\right)$.

So, with

$$g'\left(\frac{\pi}{2}\right) = -\left(\sin\left(\frac{\pi}{2}\right)\right)(1 + \sin\left(\left[cos\,(\frac{\pi}{2})\right]^2\right)\Big) = -1$$

And,

$$g\left(\frac{\pi}{2}\right) = \int_0^{\cos(\frac{\pi}{2})} [1 + \sin(t^2)]dt = \int_0^0 [1 + \sin(t^2)]dt = 0$$

We have,

$$f'\left(\frac{\pi}{2}\right) = \frac{g'\left(\frac{\pi}{2}\right)}{\sqrt{1 + \left[g\left(\frac{\pi}{2}\right)\right]^3}} = \frac{(-1)}{1} = -1$$

So, once again, by using the FToCp1, I just solved the case.

$$\boxed{f'\left(\frac{\pi}{2}\right) = -1}$$

7. Evaluate $L = \lim_{x \to 0} \frac{1}{x} \int_0^x (1 - \tan(2t))^{\frac{1}{t}} \, dt$

We have,

$$L = \lim_{x \to 0} \frac{1}{x} \int_0^x (1 - \tan(2t)^{\frac{1}{t}}) dt = \lim_{x \to 0} \frac{\int_0^x (1 - \tan(2\mathrm{x}))^{\frac{1}{t}} \, dt}{x} (=) \lim_{x \to 0} \frac{0}{0}$$

This limit is an indeterminant form. So, in order to solve for this limit, we have to use L'Hospital's Rule.

$$L = \lim_{x \to 0} \frac{\int_0^x (1 - \tan(2t))^{\frac{1}{t}} dt}{x} = \lim_{x \to 0} \frac{\frac{d}{dx} \int_0^x (1 - \tan(2t))^{\frac{1}{t}} \, dt}{1} = \lim_{x \to 0} \frac{d}{dx} \int_0^x (1 - \tan(2t))^{\frac{1}{t}} \, dt$$

Let find the derivative of the integral first. By the FToCp1,

$$\frac{d}{dx} \int_0^x (t - \tan(2t))^{1/t} dt = 1 - \tan\,(2x)^{1/x}$$

Therefore,

$$L = \lim_{x \to 0} \frac{d}{dx} \int_0^x (1 - \tan(2t))^{\frac{1}{t}} \, dt = \lim_{x \to 0} (1 - \tan(2\mathrm{x}))^{\frac{1}{\mathrm{x}}} (=) 1^{\infty}$$

This limit is also in-determinant. But there is a way to do it.

$$L = \lim_{x \to 0} (1 - \tan(2\mathrm{x}))^{\frac{1}{\mathrm{x}}} = e^{\lim_{x \to 0} \left(\frac{1}{x}\right)(ln(1 - \tan(2\mathrm{x}))} (=) e^{0.\infty}$$

You gave it up already, didn't you? Well, you shouldn't have because there really is a way to do it: use L'Hosp1tal's Rule again.

$$L = e^{\lim_{x \to 0} \left(\frac{1}{x}\right)(ln(1 - \tan(2\mathrm{x}))} = e^{\lim_{x \to 0} \frac{(ln(1 - \tan(2\mathrm{x}))}{x}} = e^{\lim_{x \to 0} \frac{d}{dx}(\ln(1 - \tan(2x)))}$$

And, $\frac{d}{dx}(\ln(1 - \tan(2x)) = \frac{-2(\sec\,(2\mathrm{x}))^2}{1 - \tan\,(2x)}$

Thus,

$$\boxed{L = e^{\lim_{x \to 0} \frac{d}{dx}(\ln(1 - \tan(2x)))} = e^{\lim_{x \to 0} \frac{-2(\sec\,(2\mathrm{x}))^2}{1 - \tan\,(2x)}} = e^{-2} = (1/e)^2}$$

Chapter 7, Problem Plus, page 522

7. A function f is defined by

$$f(x) = \int_0^{\pi} \cos(t)\cos(x-t)\,dt \qquad\qquad 0 \le x \le 2\pi$$

Find the minimum value of f.

<u>Solve:</u>

Using the Trigonometry Identity for the Product of Cosines:

2 cos(A) cos(B) = cos(A+B) + cos(A-B)

We have,

$$f(x) = \int_0^{\pi} \cos(t)\cos(x-t)\,dt \;= \frac{1}{2}\int_0^{\pi} (\cos(x) + \cos\,(2t-x))dt$$

$$= \frac{1}{2}\int_0^{\pi} \cos(x)\,dt + \frac{1}{2}\int_0^{\pi} \cos\,(2t-x)dt$$

Now, be careful here, you are integrating the two integral with respect to t, not x. So, for the first integral, we have

$$\frac{1}{2}\int_0^{\pi} \cos(x)\,dt = \frac{1}{2}\cos(x)\,(\pi - 0) = \frac{\pi}{2}\cos\,(x)$$

And for the second integral, we use substitution.

With $u = 2t - x, du = 2dt$

$$\frac{1}{2}\int_0^{\pi} \cos\,(2t-x)dt = \frac{1}{4}\int_0^{\pi} \cos\,(u)du = \frac{1}{4}(\sin(2\pi - \mathrm{x}) - \sin\,(-\mathrm{x}))$$

$$= \frac{1}{4}(\sin(\mathrm{x}) - \sin(\mathrm{x})) = 0$$

So,

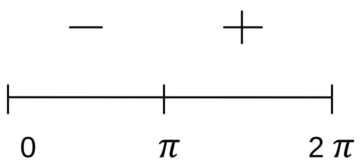
$$f(x) = \int_0^{\pi} \cos(t)\cos(x-t)\,dt \;= \frac{\pi}{2}\cos(x) + 0 = \frac{\pi}{2}\cos(x)$$

Thus,

$$f'(x) = -\frac{\pi}{2}\sin(x)$$

In order to find the minimum of f(x), we have to find where f'(x) is equal to 0. (f'(x) is always defined in the interval $0 \le x \le 2\pi$)

$$f'(x) = 0 \overset{0\le x\le 2\pi}{\Longleftrightarrow} -\frac{\pi}{2}\sin(x) = 0 \overset{0\le x\le 2\pi}{\Longleftrightarrow} x = 0, \pi, or\ 2\pi$$



We see that x is decreasing on [0, π], and increasing on [π, 2 π]. Thus, f(x) will have a min value at $x = \pi$.

Hence, the minimum value of f(x) is

$$\boxed{f(\pi) = \frac{\pi}{2}\cos(\pi) = -\frac{\pi}{2}}$$

*Glossary

*Here is The Fundamental Theorem of Calculus, Part 1for those who don't know what it is.

If f is continuous on [a, b], then the function g defined by

$$g(x) = \int_a^x f(t)dt \qquad a \le x \le b$$

is continuous on [a, b] and differentiable on (a, b), and g'(x) = f(x).