Review for Test 4
Math 125: Calculus II

Format

- The exam will contain 7-8 problems (plus or minus 1) and will last 50 minutes.
- It is a paper and pencil exam.
- You will need to show your work.
- You may use a graphing calculator. However, you may not use a symbolic calculator such as the TI-89.

Basic Content.

- The new material on the exam includes sections 7.7, 7.8, 8.1, and 8.2.
- Additionally, you are responsible for the material on Tests 1 and 2. More info to follow.
- In addition to the material covered in the class, you are responsible for all of the basic facts you have learned since kindergarten. These include the facts that Barack Obama is the President of the United States of America, $-1^2 = -1$, and that $\int_0^\pi \sin(x)\,dx = 2$.

In Studying . . .

- You should be able to recreate every proof done in class.
- You should be able to solve every example done in class.
- You should be able to solve every homework question.
- You should work through all the problems on the first exam.

A Summary of the Topics

Section 7.7: Numerical Integration

- Understand the various methods described in the section including:
  - The Trapezoidal Rule
  - The Midpoint Rule
  - Simpson's Rule
- Be able to approximate an integral using any of the given methods *by hand* up to n=8.
  - To be clear, by hand includes letting you use the calculator. I just mean to imply that some crunching will likely be required.
- Be able to bound the error using any of the above mentioned methods.
- Given an error bound, be able to find n such that the approximation is guaranteed to be within the bound (the minimum n that satisfies the criteria).
- Notes:
  - I will not ask you to calculate the derivatives needed for the error bounding.

Section 7.8: Improper Integrals

- Recognize improper integrals.
- Evaluate improper integrals using the correct notation (don't forget your limits).
- Determine convergence or divergence by using the comparison test for improper integrals.

Section 8.1: Arclength

- Calculate the arclength of a curve.

Section 8.2: Area of a Surface of Revolution

- Find the surface area of a surface of revolution (whether rotated about the x or y axis).

Practice Problems:

1.) Find the smallest n such that you can approximate $\int_0^1 e^{2x}\,dx$ within 0.00001 using Simpson's Rule.

Note: n must be an even number. Hint: You can verify n by approximating the integral with S_n and calculating the error.

$$f(x) = e^{2x}$$

$$f^{(4)}(x) = 16e^{2x}$$

max of $f^{(4)}(x)$ on $[0,1]$ is $16e^2$

$$|E_S| \le 0.00001 \le \frac{16e^2(1-0)^5}{180n^4}$$

$$\Rightarrow \quad n \le \underbrace{\sqrt[4]{\frac{16e^2}{180(0.00001)}}}_{16.0088}$$

choose $n = 18$ next largest even integer.

2.) Use the comparison theorem to determine whether the integral $\int_0^1 \frac{e^{-x}}{\sqrt{x}}\,dx$ is convergent or divergent.

$$\frac{1/e}{\sqrt{x}} \leq \frac{e^{-x}}{\sqrt{x}} \leq \frac{1}{\sqrt{x}} \quad \text{on} \quad (0,1]$$

$$\Rightarrow \int_0^1 \frac{1/e}{\sqrt{x}}\,dx \leq \int_0^1 \frac{e^{-x}}{\sqrt{x}}\,dx \leq \int_0^1 \frac{1}{\sqrt{x}}\,dx$$

since $\int_0^1 \frac{dx}{\sqrt{x}}$ converges by the p-test, we have that $\int_0^1 \frac{e^{-x}}{\sqrt{x}}\,dx$ converges by comparison.

3.) Find the length of the curve $y = \frac{x^2}{2} - \frac{\ln x}{4}$ on $2 \leq x \leq 4$

$$y' = x - \frac{1}{4x} = \frac{4x^2-1}{4x}$$

$$L = \int_2^4 \sqrt{1 + \left(\frac{4x^2-1}{4x}\right)^2}\,dx$$

$$= \int_2^4 \sqrt{1 + \frac{(4x^2)^2 - 8x^2 + 1}{16x^2}}\,dx$$

$$= \int_2^4 \sqrt{16x^4 + 8x^2 + 1}\,dx$$

$$= \int_2^4 (4x^2+1)\,dx$$

$$= \left[\frac{4}{3}x^3 + x\right]_2^4$$

$$\Rightarrow \frac{4}{3}(64-8) + 1(4-2)$$

4.) Use the Trapezoidal and Simpson's Rule to approximate $\int_0^4 \cos\left(\sqrt{x}\right)dx$ using 10 subintervals of equal width.

$y1 = \cos(\sqrt{x})$

$L_1 =$ "seq(0.4x + 0, x, 0, 10, 1)"

$L_2 =$ "$Y_1(L_1)$"

L_3 → Trapezoidal Rule $\{1, 2, 2, \ldots, 2, 1\}$

L_3 → Simpsons Rule $\{1, 4, 2, 4, \ldots, 2, 4, 1\}$

$L_4 =$ "$L_2 * L_3$"

$$T_{10} = \frac{0.4}{2}\,\mathrm{sum}(L_4) = 0.808532$$

$$S_{10} = \frac{0.4}{3}\,\mathrm{sum}(L_4) = 0.804896$$

$$\mathrm{fnint}(y_1(x), x, 0, 4) = 0.804896$$

5.) Determine whether $\int_{-\infty}^{\infty} \frac{x^2}{9+x^6}\,dx$ converges or diverges. Evaluate it if it converges.

$$\int_0^{\infty} \frac{x^2}{9+x^6}\,dx = \lim_{t\to\infty} \int_0^t \frac{x^2}{9+x^6}\,dx$$

Let $u = x^3$

$\frac{du}{3} = x^2 dx$

$$= \lim_{t\to\infty} \int_0^{t^3} \frac{1}{3} \frac{du}{9+u^2}$$

$$= \lim_{t\to\infty} \frac{1}{3}\left[\frac{1}{3}\arctan\left(\frac{u}{3}\right)\right]_0^{t^3}$$

$$= \frac{1}{9} \cdot \frac{\pi}{2}$$

$$= \frac{\pi}{18}$$

The LHS "$(-\infty, 0]$" would do the same by symmetry.

so $\int_{-\infty}^{\infty} \frac{x^2}{9+x^6}\,dx = 2 \cdot \frac{\pi}{18} = \frac{\pi}{9}$.

6.) Find the area of the surface obtained by rotating the curve $x = \frac{1}{3}\left(y^2+2\right)^{3/2}$ on $1 \le y \le 2$ about the x-axis.

$$SA = \int_c^d 2\pi y \sqrt{1+(x')^2}\, dy.$$

$$x' = \frac{1}{\cancel{3}} \cdot \frac{\cancel{3}}{\cancel{2}}(y^2+2)^{1/2} \cdot \cancel{2}y = y\sqrt{y^2+2}$$

$$SA = \int_1^2 2\pi y \sqrt{1+y^4+2y^2}\, dy$$

$$= 2\pi \int_1^2 y(y^2+1)\, dy$$

$$= 2\pi \left[\frac{y^4}{4} + \frac{y^2}{2}\right]_1^2$$

$$= 2\pi\left[(4+2) - \left(\tfrac{1}{4}+\tfrac{1}{2}\right)\right] = \frac{21\pi}{2}$$

7.) Find the area of the surface obtained by rotating the curve $y = \frac{1}{4}x^2 - \frac{1}{2}\ln(x)$ on $1 \le x \le 2$ about the y-axis.

$$SA = \int_a^b 2\pi x \sqrt{1+(y')^2}\, dx.$$

$$y' = \frac{x}{2} - \frac{1}{2x} = \frac{x^2-1}{2x}$$

$$SA = \int_1^2 2\pi x \sqrt{1 + \frac{x^4-2x^2+1}{4x^2}}\, dx$$

$$= \cancel{2}\pi \int_1^2 \frac{\cancel{x}(x^2+1)}{\cancel{2}\cancel{x}}\, dx$$

$$\cancel{= 2\pi\left[\frac{x^4}{4} + \frac{x^2}{2}\right]_1^2 = 2\pi\left[(4+2) - \left(\tfrac{1}{4}+\tfrac{1}{2}\right)\right]}$$

$$= \pi\left(\frac{x^3}{3} + x\right)\Big|_1^2 \longrightarrow \pi\left(\left(\tfrac{8}{3}+2\right) - \left(\tfrac{1}{3}+1\right)\right) = \frac{10}{3}$$