This derivation of the Fundamental Theorem of Calculus is taken from George Simmons delightful text, "Calculus with Analytic Geometry" (2nd ed.) printed by McGraw Hill in 1996. The pages come from the range 206-209.

# 6.6 THE FUNDAMENTAL THEOREM OF CALCULUS

As our main achievement so far in this chapter, we have formulated a rather complicated definition of the definite integral of a continuous function as the limit of approximating sums,

$$\int_a^b f(x)\,dx = \lim_{\max \Delta x_k \to 0} \sum_{k=1}^{n} f(x_k^*)\,\Delta x_k. \quad (1)$$

We have also considered several examples of the use of this definition in calculating the values of certain simple integrals, such as

$$\int_0^b x\,dx = \frac{b^2}{2}, \qquad \int_0^b x^2\,dx = \frac{b^3}{3}, \qquad \text{and} \qquad \int_0^b x^3\,dx = \frac{b^4}{4}. \quad (2)$$

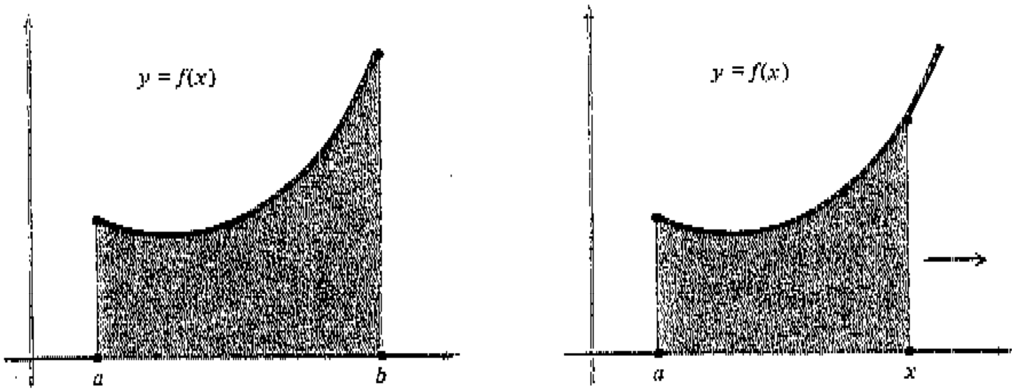


Figure 6.19

These calculations have had two purposes: to emphasize the essential nature of the integral by giving students some direct experience with approximating sums, and also to suggest the severe limitations of this method as a practical tool for evaluating integrals. Thus, for example, how can we possibly use limits of sums to find the numerical values of such complicated integrals as

$$\int_0^1 \frac{x^4\,dx}{\sqrt[3]{7+x^5}} \quad \text{and} \quad \int_1^2 \left(1+\frac{1}{x}\right)^4 \frac{dx}{x^2}? \tag{3}$$

This is clearly out of the question, so where do we go from here? What is evidently needed is a much more efficient and powerful method of computing integrals, and we find this method in the ideas of Newton and Leibniz.

The Newton-Leibniz approach to the problem of calculating the integral (1) depends on an idea that seems paradoxical at first sight. In order to solve this problem, we replace it by an apparently harder problem. Instead of asking for the *fixed* area on the left in Fig. 6.19, we ask for the *variable* area produced when the edge on the right side of the figure is considered to be moveable, so that the area is a function of $x$, as suggested on the right in Fig. 6.19. If this area function is denoted by $A(x)$, then clearly $A(a) = 0$ and $A(b)$ is the fixed area on the left in the figure. Our aim is to find an explicit formula for $A(x)$, and then to determine the desired fixed area by setting $x = b$. There are several steps in this process, which we consider separately for the sake of clarity.

STEP 1 We begin by establishing the crucial fact that

$$\frac{dA}{dx} = f(x). \tag{4}$$

This says that *the rate of change of the area A with respect to x is equal to the length of the right edge of the region.* To prove this statement, we must appeal to the definition of the derivative,

$$\frac{dA}{dx} = \lim_{\Delta x \to 0} \frac{A(x+\Delta x) - A(x)}{\Delta x}.$$

Now $A(x)$ is the area under the graph between $a$ and $x$, and $A(x + \Delta x)$ is the area between $a$ and $x + \Delta x$. Hence the numerator $A(x + \Delta x) - A(x)$ is the area between $x$ and $x + \Delta x$ (see the shaded region in Fig. 6.20). It is easy to see that this area is exactly equal to the area of a rectangle with the same base whose

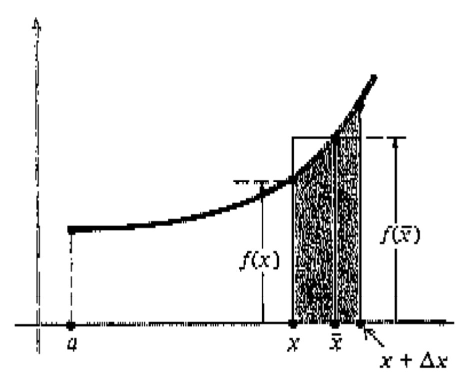


Figure 6.20

see note

height is $f(\bar{x})$, where $\bar{x}$ is a suitably chosen point between $x$ and $x + \Delta x$.* This enables us to complete the proof of (4) as follows:

$$\frac{dA}{dx} = \lim_{\Delta x \to 0} \frac{A(x + \Delta x) - A(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(\bar{x})\,\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} f(\bar{x}) = f(x),$$

since $f(x)$ is continuous. To explain the last step here in a bit more detail, we point out that $\Delta x \to 0$ is equivalent to $x + \Delta x \to x$; since $\bar{x}$ is caught between $x$ and $x + \Delta x$, we also have $\bar{x} \to x$, and the continuity of the function now yields the conclusion that $f(\bar{x}) \to f(x)$.

STEP 2 Equation (4) makes it possible for us to achieve our goal of finding a formula for the area function $A(x)$. The reasoning goes this way. By (4), $A(x)$ is one of the antiderivatives of $f(x)$. But if $F(x)$ is *any* antiderivative of $f(x)$, then we know from Chapter 5 that

$$A(x) = F(x) + c \tag{5}$$

for some value of the constant $c$. To determine $c$, we put $x = a$ in (5) and obtain $A(a) = F(a) + c$; but since $A(a) = 0$, this yields $c = -F(a)$. Therefore

$$A(x) = F(x) - F(a) \tag{6}$$

is the desired formula.

STEP 3 All that remains is to observe that

$$\int_a^b f(x)\,dx = A(b) = F(b) - F(a),$$

by (6) and the meaning of $A(x)$.

We summarize our conclusions by formally stating the Fundamental Theorem of Calculus:

*If $f(x)$ is continuous on a closed interval $[a, b]$, and if $F(x)$ is any antiderivative of $f(x)$, so that $(d/dx)\,F(x) = f(x)$ or equivalently*

$$\int f(x)\,dx = F(x), \tag{7}$$

*then*

$$\int_a^b f(x)\,dx = F(b) - F(a). \tag{8}$$

This theorem transforms the difficult problem of evaluating definite integrals by calculating limits of sums into the much easier problem of finding antiderivatives. To find the value of $\int_a^b f(x)\,dx$, we therefore no longer have to think about sums at all; we merely find an antiderivative $F(x)$ in any way we can—by in-

*When this statement is expressed in formal language, it is called the *First Mean Value Theorem of Integral Calculus*. Loosely speaking, if the top of the rectangle is at just the right level, then the part of the area protruding above it exactly balances the deficiency below it.

spection, routine calculation, ingenious calculation, or looking it up in a book—and then compute the number $F(b) - F(a)$.

For instance, in Section 6.5 we used a good deal of algebraic ingenuity to obtain the formulas (2). Now, with the aid of the Fundamental Theorem, we see these formulas as obvious consequences of the following simple facts:

$$\int x\,dx = \frac{x^2}{2}, \qquad \int x^2\,dx = \frac{x^3}{3}, \qquad \text{and} \qquad \int x^3\,dx = \frac{x^4}{4}.$$

More generally, for any exponent $n > 0$ we clearly have

$$\int_a^b x^n\,dx = \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1}, \qquad \text{because} \qquad \int x^n\,dx = \frac{x^{n+1}}{n+1}.$$

**Remark 1** In the process of working problems, it is often convenient to use the *bracket symbol,*

$$F(x)\Big]_a^b = F(b) - F(a), \tag{9}$$

which is read "$F(x)$ bracket $a$, $b$." This symbol means exactly what (9) says it does: To find its value, we write the value of $F(x)$ when $x$ has the upper value $b$, and subtract the value of $F(x)$ when $x$ has the lower value $a$. For example, $x^2\Big]_3^4 = 4^2 - 3^2 = 16 - 9 = 7$. By using this notation, (8) can be written in the form

$$\int_a^b f(x)\,dx = F(x)\Big]_a^b.$$

**Remark 2** It should be clear from this discussion that *any* antiderivative of $f(x)$ will do in (8). In case students are in doubt about this, they should recall that if $F(x)$ is one antiderivative, then any other can be obtained by adding a suitable constant $c$ to form $F(x) + c$; and since

$$F(x) + c\Big]_a^b = [F(b) + c] - [F(a) + c] = F(b) - F(a),$$

the constant $c$ has no effect on the result. We may therefore ignore constants of integration when finding antiderivatives for the purpose of computing definite integrals. (Nevertheless, these constants of integration remain indispensable when we are working with differential equations, as we saw in Section 5.4.)

**Example 1** Evaluate each of the following definite integrals:

(a) $\int_{-1}^2 x^4\,dx$; (b) $\int_1^{16} \frac{dx}{\sqrt{x}}$; (c) $\int_8^{27} \sqrt[3]{x}\,dx$; (d) $\int_{13}^{14} (x-13)^{10}\,dx$.

*Solution* In each case an antiderivative is easy to find by inspection:

$$\text{(a)} \int_{-1}^2 x^4\,dx = \frac{1}{5}x^5\Big]_{-1}^2 = \frac{1}{5}[32 - (-1)] = \frac{33}{5};$$

$$\text{(b)} \int_1^{16} \frac{dx}{\sqrt{x}} = 2\sqrt{x}\Big]_1^{16} = 2(4-1) = 6;$$