Test 3
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Math 220

Name: key.

Mathematics is a game played according to certain simple rules with meaningless marks on paper.

David Hilbert
1862 - 1943 (Prussian mathematician)

No work = no credit

$\det\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\right) = -2$ $\qquad \det(I) = 1 \qquad \det(A^{-1}) = -\frac{1}{2}$

Warm-ups (1 pt each): $\cancel{A\theta} =$ ______ $\qquad \theta^T \cdot \theta =$ ______ $\qquad \theta \cdot \theta^T =$ ______

1.) (1 pt) According to Hilbert, how much transcendent or intrinsic meaning is there in mathematics? (See above). Answer using complete English sentences.

There is no meaning. Math is just a game.

2.) (10 pts) Find the QR factorization of $A = \begin{bmatrix} 1 & 0 \\ 1 & 2 \\ 0 & 1 \\ 2 & 2 \end{bmatrix}$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix} \qquad \|\vec{v}_1\| = \sqrt{6} \qquad \vec{u}_1 = \begin{bmatrix} 1/\sqrt{6} \\ 1/\sqrt{6} \\ 0 \\ 2/\sqrt{6} \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 2 \end{bmatrix} \qquad v_2^{\perp} = v_2 - \underbrace{\langle \vec{u}_1 \cdot \vec{v}_2 \rangle}_{\sqrt{6}} \vec{u}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\|\vec{v}_2^{\perp}\| = \sqrt{3} \qquad \vec{u}_2 = \begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ 1/\sqrt{3} \\ 0 \end{bmatrix}$$

$$A = QR = \begin{bmatrix} 1/\sqrt{6} & -1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & 1/\sqrt{3} \\ 2/\sqrt{6} & 0 \end{bmatrix} \begin{bmatrix} \sqrt{6} & \sqrt{6} \\ 0 & \sqrt{3} \end{bmatrix}$$

3.) (10 pts) Consider the experimental observations given in the following table:

t	1	4	8	11
y	1	2	4	5

Find the least-squares linear ($y = mt + b$) fit to the data using techniques developed in linear algebra. Give exact values.

$$A = \begin{array}{c} \begin{array}{cc} m & b \end{array} \\ \begin{bmatrix} 1 & 1 \\ 4 & 1 \\ 8 & 1 \\ 11 & 1 \end{bmatrix} \end{array} \qquad \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 4 \\ 5 \end{bmatrix}$$

5pts.

Solve $A\vec{x} = \vec{b}$ $\Rightarrow$ $x^* = (A^T A)^{-1} A^T \vec{b}$

$$= \begin{bmatrix} 12/29 \\ 15/29 \end{bmatrix} \approx \begin{bmatrix} 0.4137 \\ 0.5172 \end{bmatrix}$$

$$y = \frac{12}{29} t + \frac{15}{29}$$

Find the magnitude of the minimum error vector.

$\vec{error} = \vec{b} - A\vec{x}^* = [0.069 \quad -0.17 \quad 0.17 \quad -0.069]^T$

and $\|\vec{error}\| \approx \sqrt{0.06897}$

or exactly $\sqrt{\frac{2}{29}}$.

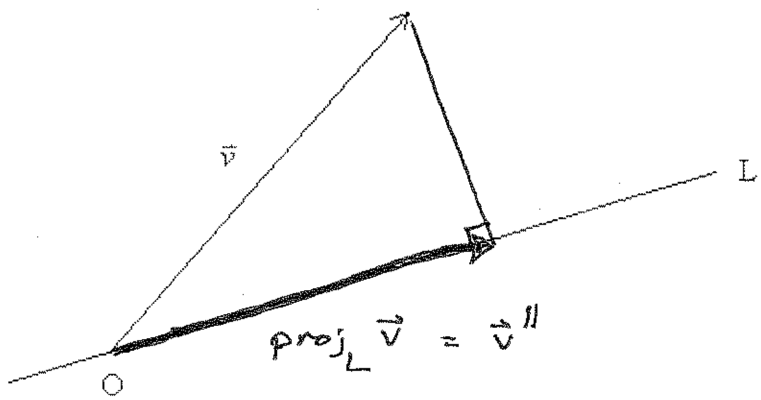
4.) (5 pts) How do you determine if a matrix A is orthogonal?

method 1: $A^T A = I$ iff A is orthogonal.

method 2: dot all cols w/ eachother. If A is orthogonal, all ~~cols~~ dot products are zero but where a vector is dotted w/ itself in which case it is 1.

5.) (5 pts) Consider the sketch below.

(a.) Clearly and carefully draw and label the orthogonal projection of $\vec{v}$ onto the line L.



(b.) Explain ~~in words~~ how you would find it given some vector $\vec{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ and the equation of the line L: $c_1 x + c_2 y = 0$.

(1) find the direction of L:

$$\vec{w} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

(2) find a unit vector parallel to $\vec{w}$: $\vec{u} = \frac{\vec{w}}{\|\vec{w}\|}$

(3) find $\text{proj}_L \vec{v} = (\vec{u} \cdot \vec{v})\vec{u}$

6.) (5 pts) If $A = QR$ is a QR factorization, prove $A^T A$ equals $R^T R$.

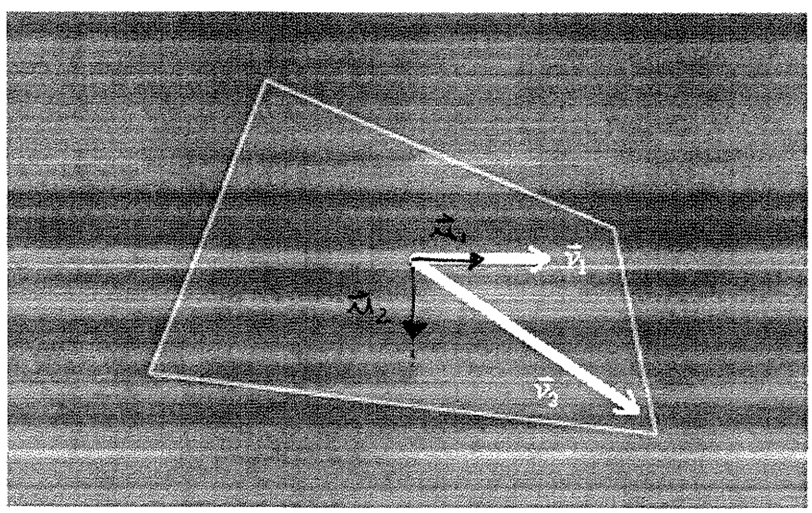
$\square$ proof.

Assume A has the QR factorization $A=QR$

$$\Rightarrow A^T A = (QR)^T (QR)$$
$$= R^T Q^T Q R$$
$$= R^T I R$$
$$= R^T R$$

QED.

7.) (5 pts) Suppose you are given two vectors $\vec{v}_1$ and $\vec{v}_2$ in $\mathbb{R}^3$ below and told to use Gram-Schmidt to generate an orthogonal basis $\{\vec{u}_1, \vec{u}_2\}$ spanning the plane. Clearly and carefully sketch and label these vectors given that $\|\vec{v}_1\| = 2$.



8.) (10 pts) Use the determinant to find out for which values of the constant λ the matrix $A - \lambda I$ fails to be invertible.

$$A = \begin{bmatrix} 3 & 5 & 6 \\ 0 & 4 & 2 \\ 0 & 2 & 7 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} 3-\lambda & 5 & 6 \\ 0 & 4-\lambda & 2 \\ 0 & 2 & 7-\lambda \end{bmatrix}$$

$$\det(A - \lambda I) = (3-\lambda) \begin{vmatrix} 4-\lambda & 2 \\ 2 & 7-\lambda \end{vmatrix}$$

$$= (3-\lambda)\left[\underbrace{(4-\lambda)(7-\lambda)}_{28 - 11\lambda + \lambda^2} - 4\right]$$

$$= (3-\lambda)(\lambda^2 - 11\lambda + 24)$$

$$= (3-\lambda)(\lambda - 3)(\lambda - 8)$$

$$= -\lambda^3 + 14\lambda^2 - 57\lambda + 72.$$

A is not invertible when $\lambda = 3$ and $\lambda = 8$

9.) ($\cancel{10}$ 5 pts) What are two geometric interpretations for $\det\left(\overset{A}{\begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix}}\right) = 5$?

(1) The area of the parallelogram determined by the cols of A.

(2) The expansion factor of the linear trans. A.