5.2: Gram-Schmidt Orthogonalization and QR Factorization

<u>Concept</u>: use a given basis $\vec{v}_1, \dots, \vec{v}_m$ to construct an orthonormal basis $\vec{u}_1, \dots, \vec{u}_m$

<u>ex 1</u>: $\vec{v}_1 = \begin{bmatrix} 2 \\ 2 \\ -2 \\ 2 \end{bmatrix} \Rightarrow \vec{u}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$

<u>ex 2</u>: $\vec{v}_1 = \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}$ & $\vec{v}_2 = \begin{bmatrix} 25 \\ 0 \\ -25 \end{bmatrix}$

$\vec{u}_1 = \begin{bmatrix} 4/5 \\ 0 \\ 3/5 \end{bmatrix}$

$$\vec{v}_2^{\perp} = \vec{v}_2 - \underbrace{(\vec{u}_1 \cdot \vec{v}_2)}_{5}\,\vec{u}_1 = \begin{bmatrix} 25 \\ 0 \\ -25 \end{bmatrix} - \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 21 \\ 0 \\ -28 \end{bmatrix}$$

and $\|\vec{v}_2^{\perp}\| = 35$ so $\vec{u}_2 = \begin{bmatrix} 3/5 \\ 0 \\ -4/5 \end{bmatrix}$

$\boxed{\begin{matrix}5.2\\ 2/3\end{matrix}}$

<u>ex 3:</u> $\vec{v}_1 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}1\\0\\0\\1\end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix}0\\2\\1\\-1\end{bmatrix}$

$$\vec{u}_1 = \begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix}$$

$$\vec{v}_2^{\perp} = \begin{bmatrix}1\\0\\0\\1\end{bmatrix} - \overset{(u_1 \cdot v_2)}{1} \begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix} = \begin{bmatrix}1/2\\-1/2\\-1/2\\1/2\end{bmatrix}$$

and $\|v_2^{\perp}\| = 1$ so $\vec{v}_2^{\perp} = \vec{u}_2$

$$\vec{v}_3^{\perp} = \begin{bmatrix}0\\2\\1\\-1\end{bmatrix} - \overset{(u_1 \cdot v_3)}{1} \begin{bmatrix}1/2\\1/2\\1/2\\1/2\end{bmatrix} + \overset{(u_2 \cdot v_3)}{(+2)} \begin{bmatrix}1/2\\-1/2\\-1/2\\1/2\end{bmatrix} = \begin{bmatrix}1/2\\+1/2\\-1/2\\-1/2\end{bmatrix}$$

and $\|\vec{v}_3^{\perp}\| = \cancel{\tfrac{5}{2}}\,1$ so $\vec{u}_3 = \cancel{\begin{bmatrix}1/\sqrt{5}\\-2/\sqrt{5}\\-2/\sqrt{5}\\1/\sqrt{5}\end{bmatrix}}\ \vec{v}_3^{\perp}$

$$M = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow \|v_1\| = 2 \quad \text{and} \quad \vec{u}_1 = \begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix}$$

$$\vec{v}_2^{\perp} = \vec{v}_2 - \vec{v}_2^{\parallel} = \vec{v}_2 - (\vec{u}_1 \cdot \vec{v}_2)\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} - (1)\begin{bmatrix} 1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$$

$$\vec{u}_2 = \frac{\vec{v}_2^{\perp}}{\underbrace{\|\vec{v}_2^{\perp}\|}_{1}} = \vec{v}_2^{\perp}$$

$$\vec{v}_3^{\perp} = \vec{v}_3 - (\vec{u}_1 \cdot \vec{v}_3)\vec{u}_1 - (\vec{u}_2 \cdot \vec{v}_3)\vec{u}_2 = \vec{v}_3 - (1)\vec{u}_1 - (-2)\vec{u}_2 = \begin{bmatrix} [illegible] \\ 1/2 \\ -1/[illegible] \\ -1[illegible] \end{bmatrix}$$

$$\vec{u}_3 = \frac{\vec{v}_3^{\perp}}{\underbrace{\|\vec{v}_3^{\perp}\|}_{1}}$$

and so $M = QR$ where

$$Q = \begin{bmatrix} 1/2 & 1/2 & 1/2 \\ 1/2 & -1/2 & 1/2 \\ 1/2 & -1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 \end{bmatrix}_{4\times 3} \quad \text{and} \quad R = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix}_{3\times 3}$$